# EFFECTIVE CLASSES IN THE PROJECTIVIZED $k$-TH HODGE BUNDLE

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We study the classes of several loci in the projectivization of the $k$-th Hodge bundle $\mathbb{P}\left(\mathbb{E}_g^k\right)$ over $\overline{\mathcal{M}}_g$ and $\mathbb{P}\left(\mathbb{E}_{g,n}^k\right)$ over $\overline{\mathcal{M}}_{g,n}$. In particular we consider the class of the closure in $\mathbb{P}\left(\mathbb{E}_{g,n}^k\right)$ of the codimension $n$ locus where the $n$ marked points are zeros of the $k$-differential. When $n = 1$ this class was computed in [KSZ19, §4]. We compute it when $n = 2$ and provide a recursive formula for it when $n > 2$. Moreover, when $n = 1$ and $k = 1, 2$ we show its rigidity and extremality in the pseudoeffective cone. We also compute the classes of the closures in $\mathbb{P}\left(\mathbb{E}_g^k\right)$ of the loci where the $k$-differential has a zero at a Brill-Noether special point.

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*To Ron.*

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# 1 Introduction

The $k$-th Hodge bundle $\mathbb{E}_g^k$ over $\overline{\mathcal{M}}_g$ has fiber over a point $[C] \in \overline{\mathcal{M}}_g$ equal to the space of stable $k$-differentials $H^0\left(C, \omega_C^{\otimes k}\right)$. We denote the projectivization of the $k$-th Hodge bundle by $\mathbb{P}\left(\mathbb{E}_g^k\right)$. One can also consider the $k$-th Hodge bundle over $\overline{\mathcal{M}}_{g,n}$, which we denote by $\mathbb{E}_{g,n}^k$ with projectivization $\mathbb{P}\left(\mathbb{E}_{g,n}^k\right)$.

For a partition $\boldsymbol{m} = (m_1, \ldots, m_n)$ of $k(2g-2)$, let $\mathcal{H}_g^k(\boldsymbol{m})$ be the stratum in $\mathbb{E}_g^k$ where $\boldsymbol{m}$ describes the multiplicities of the zeros of the $k$-differential. The strata $\overline{\mathcal{H}}_g^k(\boldsymbol{m})$ provide a natural stratification of the complement of the zero section of $\mathbb{E}_g^k$. We denote by $\overline{\mathbb{H}}_g^k(\boldsymbol{m})$ the projectivization of the induced strata in $\mathbb{P}\left(\mathbb{E}_g^k\right)$.

We will also sometimes consider a stratum in $\mathbb{P}\left(\mathbb{E}_{g,n}^k\right)$. In this case let $\boldsymbol{m} = (m_1, \ldots, m_n)$ be a partition of some number $N$ where $n \leq N \leq k(2g-2)$. We let $\overline{\mathbb{H}}_{g,n}^k(\boldsymbol{m}) \subset \mathbb{P}\left(\mathbb{E}_{g,n}^k\right)$ be the locus where the $k$-differential has a zero of multiplicity $m_i$ at the $i$th marked point.

## Incidence loci

Consider the *incidence locus*

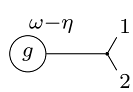
$$\mathcal{Z}_n := \overline{\left\{[C, \mu, P_1, \ldots, P_n] \in \mathbb{P}\left(\mathbb{E}_{g,n}^k\right) \mid h^0\left(C, \mathscr{O}_C\left(\mu - (P_1 + \cdots + P_n)\right)\right) \geq 1\right\}}.$$

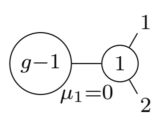
This is a non-empty codimension $n$ locus in $\mathbb{P}\left(\mathbb{E}_{g,n}^k\right)$ for $1 \leq n \leq k(2g-2)$. Note that in our notation $\mathcal{Z}_n$ is equivalent to the locus $\overline{\mathbb{H}}_{g,n}^k(1^n) \subset \mathbb{P}\left(\mathbb{E}_{g,n}^k\right)$. The incidence locus $\mathcal{Z}_{k(2g-2)} \subset \mathbb{P}\left(\mathbb{E}_{g,k(2g-2)}^k\right)$ is precisely the incidence variety compactification of the principal stratum described in [BCG$^+$19]. The class of $\mathcal{Z}_1$ was computed in [KSZ19, §4] to be $\mathcal{Z}_1 = k\omega - \eta$. In this thesis we compute the class of $\mathcal{Z}_2$ and provide a recursive formula for $\mathcal{Z}_n$ for $n > 2$.

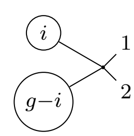
Our motivation for studying the incidence loci comes from their relation to the strata. In [FP18] the authors provide a recursive formula for the strata in $\overline{\mathcal{M}}_{g,n}$ based on a conjectural identification between the moduli space of twisted differentials and a formula of Pixton in the tautological ring of $\overline{\mathcal{M}}_{g,n}$. In [Sch18] the author extends the conjecture and recursion to the case $k > 1$. Finally, in [BHP$^+$] the authors prove the conjecture of the previous two papers and are therefore able to provide a closed formula for the strata classes in $\overline{\mathcal{M}}_{g,n}$. The strata classes have also been studied in the projectivized Hodge bundle directly. In [Sau19] the author provides a recursive formula for the strata classes in $\mathbb{PE}^1_g$.

Solving the recursion provided in Theorem 1.2 would give a formula for strata with simple zeros in $\mathbb{PE}^k_{g,n}$, under the restrictions of $k$ and $n$ described there. It would also indirectly provide a closed formula for strata with one zero of higher multiplicity, due to the presence of the loci we call $E_{\boldsymbol{I}}$ in the formula. Since the space $\mathbb{PE}^k_{g,n}$ admits natural forgetful maps to both $\overline{\mathcal{M}}_{g,n}$ and $\mathbb{PE}^k_g$, such a closed formula could be used to find the strata classes in these spaces as well.

**Theorem 1.1.** *One has*

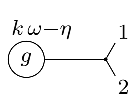
$$\mathcal{Z}_2 \equiv (\omega_1 - \eta)(\omega_2 - \eta) -$$


$$-$$


$$-\sum_{i=1}^{\lfloor g/2 \rfloor}$$


$$\in A^2\left(\mathbb{P}\left(\mathbb{E}^1_{g,2}\right)\right)$$

*and*

$$\mathcal{Z}_2 \equiv (k\,\omega_1 - \eta)(k\,\omega_2 - \eta) -$$


$$\in A^2\left(\mathbb{P}\left(\mathbb{E}^k_{g,2}\right)\right), \qquad \textit{for } k \geq 2.$$

We will explain the graph notation in Section 3.

Let $\mathcal{M}^{\mathsf{rt}}_{g,n}$ be the locus in $\overline{\mathcal{M}}_{g,n}$ of curves with rational tails; let $\pi_n\colon \mathbb{P}\left(\mathbb{E}^k_{g,n}\right) \to \mathbb{P}\left(\mathbb{E}^k_{g,n-1}\right)$ be the map obtained by forgetting the last marked point; let $\rho_n\colon \mathbb{P}\left(\mathbb{E}^k_{g,n}\right) \to$

$\mathbb{P}\left(\mathbb{E}^k_{g,1}\right)$ be the map obtained by remembering only the last marked point (and relabeling it as $P_1$). Moreover, let $\boldsymbol{P} = \{P_1, \dots, P_n\}$ and $\boldsymbol{I} \subseteq \boldsymbol{P} \setminus P_n$ where $\boldsymbol{I} \neq \emptyset$. We denote by $E_{\boldsymbol{I}} \subset \mathbb{P}\left(\mathbb{E}^k_{g,n}\right)$ the locus of genus $g$ curves with a rational tail where the rational component contains the marked points specified by $\boldsymbol{I}$ along with $P_n$, the $k$-differential on the genus $g$ component is not the $k$th power of an abelian differential, and it has a zero of order at least $|\boldsymbol{I}|$ at the node as well as zeros at all the marked points on that component. More formally, consider the gluing map

$$\mathbb{P}\left(\mathbb{E}^k_{g,\boldsymbol{P}^-}\right) \times \overline{\mathcal{M}}_{0,\boldsymbol{I}^+} \to \mathbb{P}\left(\mathbb{E}^k_{g,\boldsymbol{P}}\right)$$

where $\boldsymbol{P}^- := \boldsymbol{P} \setminus (\boldsymbol{I} \cup \{P_n\}) \cup \{Q_1\}$, $\boldsymbol{I}^+ := \boldsymbol{I} \cup \{P_n, Q_2\}$, and the map is defined by gluing points $Q_1$ and $Q_2$. The locus $E_{\boldsymbol{I}}$ is the image of $\overline{\mathbb{H}}^k_{g,n-|\boldsymbol{I}|}(|\boldsymbol{I}|, 1^{n-|\boldsymbol{I}|-1}) \times \overline{\mathcal{M}}_{0,\boldsymbol{I}^+}$ under this map where the zero of order $|\boldsymbol{I}|$ is at $Q_1$.

**Theorem 1.2.** *One has*

$$\pi_n^* \left[\mathcal{Z}_{n-1}\right] \cdot \rho_n^* \left[\mathcal{Z}_1\right] = \left[\mathcal{Z}_n\right] + \sum_{\boldsymbol{I}} |\boldsymbol{I}| [E_{\boldsymbol{I}}]$$

*in*

*i)* $A^n\left(\mathbb{P}\left(\mathbb{E}^k_{g,n}\big|_{\mathcal{M}^{\mathrm{rt}}_{g,n}}\right)\right)$ *for*

- *all* $n$ *when* $k = 1$,
- $n \leq k(2g-2) - 2$ *when* $k = 2$,
- $n \leq k(2g-2) - 1$ *when* $k \geq 3$; *and*

*ii)* $A^n\left(\mathbb{P}\left(\mathbb{E}^k_{g,n}\right)\right)$ *when* $n \leq k$.

**Effective divisors in $\mathbb{P}\left(\mathbb{E}_g^k\right)$**

$$\mathbb{W}_g^k := \left\{(C,\mu) \in \mathbb{P}\left(\mathbb{E}_g^k\right) \mid \text{zeros of } \mu \text{ include a Weierstrass point}\right\} \subset \mathbb{P}\left(\mathbb{E}_g^k\right).$$

**Theorem 1.3.** *For $k \geq 1$, one has*

$$\left[\overline{\mathbb{W}}_g^k\right] = -g(g^2-1)\eta + k(6g^2+4g+2)\lambda - k\binom{g+1}{2}\delta_0 - \sum_{i=1}^{\lfloor g/2 \rfloor} k(g+3)i(g-i)\delta_i \quad \in \mathrm{Pic}_{\mathbb{Q}}\left(\mathbb{P}\left(\mathbb{E}_g^k\right)\right).$$

In the statement, $\eta := \mathcal{O}_{\mathbb{P}\left(\mathbb{E}_g^k\right)}(-1)$, and $\lambda$, $\delta_0$, and $\delta_i$ denote the pullbacks of the respective classes from $\overline{\mathcal{M}}_g$. Next, we generalize this and consider Brill-Noether type divisors in $\mathbb{P}\left(\mathbb{E}_g^k\right)$; that is, divisors obtained by requiring the support of the class of the $k$-differential to contain a Brill-Noether special point. For a smooth algebraic curve $C$, the variety $G_d^r(C)$ parametrizes linear series of degree $d$ and projective dimension $r$. For $\ell = (L,V)$ in $G_d^r(C)$, the *vanishing sequence* of $\ell$ at a point $P$ in $C$

$$\boldsymbol{a}^\ell(P) : 0 \leq a_0 < \cdots < a_r \leq d$$

is defined as the increasing sequence of vanishing orders of sections in $V$ at $P$. Given $g, r, d$, and a sequence $\boldsymbol{a} = (a_0, \ldots, a_r)$ such that

$$\rho(g,r,d,\boldsymbol{a}) = g - (r+1)(g-d+r) - \sum_{i=0}^{r}(a_i - i) = -1,$$

define the locus $\overline{\mathbb{H}}_{g,d}^{\boldsymbol{a}}$ in $\mathbb{P}\left(\mathbb{E}_g^k\right)$ as

$$\overline{\mathbb{H}}_{g,d}^{\boldsymbol{a}} := \overline{\left\{(C,\mu) \in \mathbb{P}\left(\mathbb{E}_g^k\right) \;\middle|\; \boldsymbol{a}^\ell(P) \geq \boldsymbol{a}, \text{for some } \ell \in G_d^r(C) \text{ and } P \in \mathrm{supp}(\mu)\right\}}.$$

We show that the class of the divisorial component of $\overline{\mathbb{H}}_{g,d}^{\boldsymbol{a}}$ lies in the cone spanned by the pullback of the Brill-Noether divisor class $\mathcal{BN}_g$ from $\overline{\mathcal{M}}_g$ and the divisor class

$\left[\overline{\mathbb{W}}_g^k\right]$ from Theorem 1.3. This is analogous to the result from [EH89] for pointed Brill-Noether divisors in $\overline{\mathcal{M}}_{g,1}$.

**Theorem 1.4.** *For $k \geq 1$ and $g \geq 3$, the divisorial class $\left[\overline{\mathbb{H}}_{g,d}^{\boldsymbol{a}}\right]$ is equal to*

$$\left[\overline{\mathbb{H}}_{g,d}^{\boldsymbol{a}}\right] = 2k(g-1)\mu\,\mathcal{BN}_g + \nu\left[\overline{\mathbb{W}}_g^k\right] \qquad \text{in } \operatorname{Pic}_{\mathbb{Q}}\left(\mathbb{P}\left(\mathbb{E}_g^k\right)\right).$$

*where $\mu, \nu \in \mathbb{Q}_{\geq 0}$.*

Explicit formulae for $\mu$ and $\nu$ were computed in [FT16] and we record them in Section 4.1.

## Rigidity and extremality results

An effective cycle class $E$ in the numerical group $N^d\left(\mathbb{P}\left(\mathbb{E}_{g,n}^k\right)\right)$ of codimension $d$ cycles on $\mathbb{P}\left(\mathbb{E}_{g,n}^k\right)$ is called *extremal* if $E = E_1 + E_2$ in $N^d\left(\mathbb{P}\left(\mathbb{E}_{g,n}^k\right)\right)$ for two effective cycle classes $E_1$ and $E_2$ implies that both $E_1$ and $E_2$ are proportional to $E$. An effective cycle class $E$ is called *rigid* if any effective cycle with class $mE$ is supported on the support of $E$.

We are able to establish some rigidity and extremality results.

**Theorem 1.5.** *The divisor $\overline{\mathbb{H}}_g^1(2, 1^{2g-4}) \subset \mathbb{P}\left(\mathbb{E}_g^1\right)$ is rigid and extremal in $\overline{\operatorname{Eff}}^1\left(\mathbb{P}\left(\mathbb{E}_g^1\right)\right)$. The divisor $\overline{\mathbb{H}}_g^2(2, 1^{4g-6}) \subset \mathbb{P}\left(\mathbb{E}_g^2\right)$ is rigid and extremal in $\overline{\operatorname{Eff}}^1\left(\mathbb{P}\left(\mathbb{E}_g^2\right)\right)$.*

**Theorem 1.6.** *The class of $\mathcal{Z}_1$ is rigid and extremal in $\overline{\operatorname{Eff}}^1\left(\mathbb{P}\left(\mathbb{E}_{g,1}^k\right)\right)$, for $k \in \{1, 2\}$.*

We prove the above theorems by making use of Teichmüller curves, which are only defined when $k = 1, 2$. Using a recursive argument we can show that $\mathcal{Z}_n$ for $n > 2$ are rigid and extremal in $\operatorname{Eff}^n\left(\mathbb{P}\left(\mathbb{E}_{g,n}^k\right)\right)$ conditioned on the rigidity and extremality of $\mathcal{Z}_{n-1}$. This holds even for $k > 2$.

**Proposition 1.7.** *For $n > 2$, if $[\mathcal{Z}_{n-1}]$ is rigid and extremal in* $\mathrm{Eff}^{n-1}\left(\mathbb{P}\left(\mathbb{E}^k_{g,n-1}\right)\right)$*, then $[\mathcal{Z}_n]$ is rigid and extremal in* $\mathrm{Eff}^n\left(\mathbb{P}\left(\mathbb{E}^k_{g,n}\right)\right)$*, provided that $\mathcal{Z}_n$ is non-empty, i.e., $n \leq k(2g-2)$.*

# 2 Background

## 2.1 The Picard group of $\mathbb{P}\left(\mathbb{E}^k_g\right)$ and $\mathbb{P}\left(\mathbb{E}^k_{g,n}\right)$

We know that

$$\mathrm{Pic}(\mathbb{P}\left(\mathbb{E}^k_g\right)) \otimes \mathbb{Q} = \langle \eta, \lambda, \delta_0, \ldots, \delta_{\lfloor g/2 \rfloor} \rangle$$

where $\eta := \mathcal{O}_{\mathbb{P}\left(\mathbb{E}^k_g\right)}(-1)$ and the remaining classes are the pullbacks from $\overline{\mathcal{M}}_g$ [KZ11, Lemma 1]. Let $\pi : \overline{\mathcal{C}}_g \to \overline{\mathcal{M}}_g$ be the universal curve and $\omega_\pi$ the relative dualizing sheaf. The $k$th Hodge bundle $\mathbb{E}^k_g = \pi_*\left(\omega_\pi^{\otimes k}\right)$. The class $\lambda := c_1(\mathbb{E}^1_g)$, i.e., the first Chern class of the Hodge bundle. The class $\delta_0$ is the class of the closure of the locus of irreducible curves with a node and the class $\delta_i$ is the class of the closure of the locus of one-nodal curves with a genus $i$ component and a genus $g-i$ component. We may also replace $\lambda$ in the above expression with the pullback of $\kappa$ from $\overline{\mathcal{M}}_g$, where $\kappa = \pi_* c_1^2(\omega_\pi)$. This is because $\kappa = 12\lambda - \delta$, where $\delta$ is the total boundary class.

Let $S \subseteq \{1, \ldots, n\}$. We have that

$$\mathrm{Pic}(\overline{\mathcal{M}}_{g,n}) \otimes \mathbb{Q} = \langle \lambda, \psi_1, \ldots, \psi_n, \delta_0, \delta_{i;S} \rangle$$

where $0 \leq i \leq \lfloor g/2 \rfloor$. We denote by $\psi_i$ the first Chern class of the cotangent bundle on $\overline{\mathcal{M}}_{g,n}$ at the $i$th marked point. This means that on a family $\pi : \mathcal{X} \to B$, the divisor $\psi_i$ takes the value $-\pi_*(S_i^2)$ where $S_i$ marks the $i$th section. We denote by $\delta_{i;S}$ the class of the divisor whose general points parameterize one-nodal curves whose genus

$i$ component contains the markings labelled by the subset $S$. Note that if $i = 0$, then $|S| \geq 2$. Furthermore, if $i = g/2$, we force $1 \in S$. For more details, see [AC87]. We will also often make use of the classes $\omega_i, 1 \leq i \leq n$, defined as follows. If $\omega_\pi$ denotes the relative dualizing sheaf of $\pi : \overline{\mathcal{C}}_g \to \overline{\mathcal{M}}_g$ and $\rho_i : \overline{\mathcal{M}}_{g,n} \to \overline{\mathcal{M}}_{g,1}$ is the morphism forgetting all but the $i$th marked point, we let $\omega_i = \rho_i^* \omega_\pi$. When $n = 1$, $\omega_1 = \psi_1$. More generally, if $\pi_i : \overline{\mathcal{M}}_{g,n} \to \overline{\mathcal{M}}_{g,n-1}$ is the morphism forgetting the $i$th point, we have the following theorem.

**Theorem 2.1** ([Log03, Theorem 2.3])**.** *One has*

$$\begin{aligned}
\pi_n^* \lambda &= \lambda \\
\pi_n^* \delta_0 &= \delta_0 \\
\pi_n^* \omega_i &= \omega_i \\
\pi_n^* \psi_i &= \psi_i - \delta_{0;\{i,n\}} \\
\pi_n^* \delta_{i;S} &= \delta_{i;S} + \delta_{i;S \cup \{n\}}.
\end{aligned}$$

*except that* $\pi_1^* \delta_{g/2;\emptyset} = \delta_{g/2,\emptyset}$ *for* $n = 1$.

One important consequence of this is that

$$\psi_i = \omega_i + \sum_{i \in S \subset \{1,\ldots,n\}} \delta_{0;S}.$$

We note that by the above discussion,

$$\mathrm{Pic}(\mathbb{P}\left(\mathbb{E}_{g,n}^k\right)) \otimes \mathbb{Q} = \langle \eta, \lambda, \psi_1, \ldots, \psi_n, \delta_0, \delta_{i;S} \rangle.$$

## 2.2 The incidence variety compactification of the strata

We will also regularly make use of the incidence variety compactification of the strata. For a partition $\mu = (m_1, \ldots, m_n)$ of $k(2g-2)$, define

$$\mathcal{P}^k(\mu) := \left\{(C, \xi, z_1, \ldots, z_n) \in \mathbb{P}\left(\mathbb{E}^k_{g,n}\right) \mid \operatorname{div} \xi = \sum_{i=1}^n m_i z_i\right\}.$$

The incidence variety compactification $\overline{\mathcal{P}}^k(\mu)$ is defined to be the closure of $\mathcal{P}^k(\mu)$ inside $\mathbb{P}\left(\mathbb{E}^k_{g,n}\right)$. In [BCG⁺19] the authors give a full characterization of elements in the boundary, building off work in [BCG⁺18] for the case $k = 1$. We refer the reader to [BCG⁺19] for the details of this characterization. Note that we will use the notation $\overline{\mathcal{P}}^k(\mu)$ for the incidence variety compactification of a stratum, whereas in [BCG⁺19] it is denoted $\mathbb{P}\Omega^k\overline{\mathcal{M}}^{\text{inc}}_{g,n}(\mu)$.

Finally, we record here a relation involving $\eta$. The reader can see [Che20] for more details. Let $\pi : \mathcal{X} \to B$ be a one-parameter family of pointed stable $k$-differentials in $\overline{\mathcal{P}}^k(\mu)$ whose generic fiber is smooth. If $\mathcal{X}$ is singular we replace it by its minimal resolution. Let $S_1, \ldots, S_n$ be the distinct sections which mark the zeros and poles of the $k$-differentials parameterized by this family and let $\omega$ be the relative dualizing line bundle class of $\pi$. Moreover, let $V$ be the union of the irreducible components where the parameterized $k$-differentials are identically zero. Then, since the fiber of $\pi^*\eta$ over a pointed stable $k$-differential is the pointed stable $k$-differential itself, which has zeros or poles along the $S_i$ with multiplicity $m_i$ and zeros along $V$, we have a relation of divisor classes in $\mathcal{X}$

$$\pi^*\eta = \omega^{\otimes k} - \sum_{i=1}^n m_i S_i - V.$$

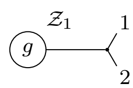
# 3 The class of the incidence locus $\mathcal{Z}_n$

As mentioned in the introduction, the class of $\mathcal{Z}_1$ was computed in [KSZ19]:

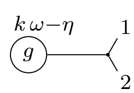
**Lemma 3.1** ([KSZ19, §4]). *One has* $\mathcal{Z}_1 \equiv k\,\omega - \eta$ *in* $\operatorname{Pic}\left(\mathbb{P}\left(\mathbb{E}^k_{g,1}\right)\right)$*, for* $k \geq 1$.

## 3.1 The class $\mathcal{Z}_2$

To describe $\mathcal{Z}_2$, we need the following loci in $\mathbb{P}\left(\mathbb{E}^k_{g,2}\right)$: let




be the push-forward of $\mathcal{Z}_1 \subset \mathbb{P}\left(\mathbb{E}^k_{g,1}\right)$ via the map $\mathbb{P}\left(\mathbb{E}^k_{g,1}\right) \to \mathbb{P}\left(\mathbb{E}^k_{g,2}\right)$ obtained by attaching at the marked point of an element in $\mathbb{P}\left(\mathbb{E}^k_{g,1}\right)$ a rational tail $R$ containing the two marked points $P_1, P_2$. Namely, $(C, \mu, Q) \mapsto (C \cup_Q R, \mu, P_1, P_2)$; the class of stable $k$-differentials $\mu$ on the curve $C$ gives rise to a class of stable $k$-differentials (still denoted $\mu$) on $C \cup_Q R$ whose aspect on the rational component is zero. The class of this locus, denoted below as



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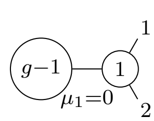
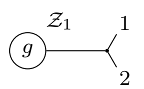
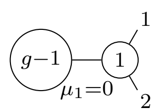
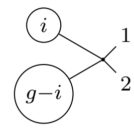
is the push-forward of the class $[\mathcal{Z}_1] = k\,\omega - \eta$ (Lemma 3.1) via the map $\mathbb{P}\left(\mathbb{E}^k_{g,1}\right) \to \mathbb{P}\left(\mathbb{E}^k_{g,2}\right)$ given above.

Moreover, let



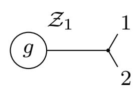

be the locus consisting of curves with an elliptic tail containing both marked points together with a class of stable $k$-differentials whose aspect on the elliptic component is zero. This locus in $\mathbb{P}\left(\mathbb{E}^k_{g,2}\right)$ has codimension two for $k=1$, and codimension $k+1$ for $k \geq 2$. Also, we use below the maps $\pi_i \colon \mathbb{P}\left(\mathbb{E}^k_{g,2}\right) \to \mathbb{P}\left(\mathbb{E}^k_{g,1}\right)$ obtained by forgetting the $i$-th marked point and relabeling the remaining marked point by $P_1$, for $i \in \{1,2\}$. Using this notation, we have:

**Proposition 3.2.** *(i) The intersection $\pi_1^{-1}(\mathcal{Z}_1) \cap \pi_2^{-1}(\mathcal{Z}_1)$ in $\mathbb{P}\left(\mathbb{E}^k_{g,2}\right)$ consists of the loci:*

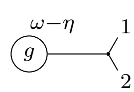
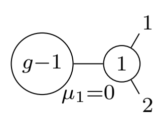
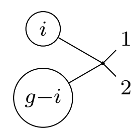
$\mathcal{Z}_2$,  ,  , *and*  *for* $i = 1, \dots, \lfloor g/2 \rfloor$ (3.1)



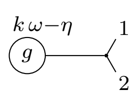
*when $k=1$, or*

$\mathcal{Z}_2$ *and*  , *when* $k \geq 2$. (3.2)

*(ii) One has*

$\pi_1^*[\mathcal{Z}_1]\cdot\pi_2^*[\mathcal{Z}_1] = [\mathcal{Z}_2] +$  $+$  $+ \sum_{i=1}^{\lfloor g/2 \rfloor}$  $\in A^2\left(\mathbb{P}\left(\mathbb{E}^1_{g,2}\right)\right)$

*and*

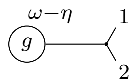
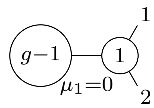
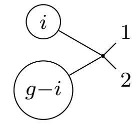
$\pi_1^*[\mathcal{Z}_1]\cdot\pi_2^*[\mathcal{Z}_1] = [\mathcal{Z}_2] +$  $\in A^2\left(\mathbb{P}\left(\mathbb{E}^k_{g,2}\right)\right)$, *when* $k \geq 2$.

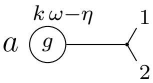
*Proof.* $(i)$ It is clear that the two loci listed in (3.2) are in $\pi_1^{-1}(\mathcal{Z}_1) \cap \pi_2^{-1}(\mathcal{Z}_1)$ for all $k \geq 1$. To see that the same is true for the two remaining types of loci in (3.1) when $k=1$, it is useful to view $\mathcal{Z}_1 \subset \mathbb{P}\left(\mathbb{E}^k_{g,1}\right)$ and $\mathcal{Z}_2 \subset \mathbb{P}\left(\mathbb{E}^k_{g,2}\right)$ as images of the incidence variety compactification of the principal stratum in $\mathbb{P}\left(\mathbb{E}^k_{g,2g-2}\right)$ under the morphisms $\mathbb{P}\left(\mathbb{E}^k_{g,k(2g-2)}\right) \to \mathbb{P}\left(\mathbb{E}^k_{g,1}\right)$ and $\mathbb{P}\left(\mathbb{E}^k_{g,k(2g-2)}\right) \to \mathbb{P}\left(\mathbb{E}^k_{g,2}\right)$ obtained by forgetting all but the first one or first two marked points, respectively. This provides a description of the boundaries of $\mathcal{Z}_1$ and $\mathcal{Z}_2$. When $k=1$, consider the third locus listed in (3.1) and fix a general element in it, with elliptic tail denoted $E$. Since we can always find a twisted abelian differential whose aspect on $E$ has a pole of order 2 at the node and a zero at the marked point $P_i$, the image of the third locus under $\pi_i$ is in $\mathcal{Z}_1$, for $i \in \{1,2\}$. The aspect of such a twisted abelian differential on $E$ does not vanish generically at both marked points, hence such a locus is not in $\mathcal{Z}_2$. However when $k \geq 2$, such a locus has codimension $k+1$ and is contained in $\mathcal{Z}_2$. Indeed, one can always find a twisted $k$-differential with pole of order at least $k+1$ at the node and vanishing at *both* marked points. For the remaining type of loci in (3.1), consider the rational component of a general element. The aspects of a general abelian differential on the two components of positive genus is regular and non-zero at the nodes. Given $j \in \{1,2\}$, the global residue condition is satisfied by a twisted abelian differential whose aspect on the rational component has a simple zero at the point $P_j$ and at another smooth point, and poles of order 2 at both nodes. This implies that the image of the last type of loci in (3.1) via $\pi_j$ is in $\mathcal{Z}_1$ when $k=1$, for each $j \in \{1,2\}$. However when $k \geq 2$, the aspects of a general stable $k$-differential on the two components of positive genus have poles of order $k$ at the nodes, hence the aspect on the rational component cannot have any zeros. It follows that the last loci in (3.1) do not lie in the intersection $\pi_1^{-1}(\mathcal{Z}_1) \cap \pi_2^{-1}(\mathcal{Z}_1)$ for $k \geq 2$.

Next, we show that the loci in (3.1) and (3.2) are the *only* ones in $\pi_1^{-1}(\mathcal{Z}_1) \cap \pi_2^{-1}(\mathcal{Z}_1)$ for $k=1$ and $k \geq 2$, respectively. On the locus of smooth curves, by definition $\mathcal{Z}_2$ is the only component in $\pi_1^{-1}(\mathcal{Z}_1) \cap \pi_2^{-1}(\mathcal{Z}_1)$ for all $k \geq 1$. To detect additional contributions on $\mathbb{P}\left(\mathbb{E}^k_{g,2}\right)$, it is sufficient to analyze all possible components whose general element $(C, \mu, P_1, P_2)$ satisfies: (a) $C$ is one-nodal, or (b) $C$ is two-nodal. For the one-nodal case, we show that if $(C, \mu, P_1, P_2)$ is both in the inverse image of a boundary divisor from $\overline{\mathcal{M}}_{g,2}$ and in $\pi_1^{-1}(\mathcal{Z}_1) \cap \pi_2^{-1}(\mathcal{Z}_1)$, then it is in one of the first three loci in (3.1) when $k = 1$, or in one of the loci in (3.2) when $k \geq 2$. In the case of one non-disconnecting node, such an element $(C, \mu, P_1, P_2)$ will need to have both marked points $P_1$ and $P_2$ in the support of $\mu$. This means that $(C, \mu, P_1, P_2)$ is in fact in $\mathcal{Z}_2$. In the case $C$ is one-nodal with a rational tail containing the two marked points, $\mu$ is necessarily the class of a stable $k$-differential vanishing at the nodal point on the genus $g$ component of $C$. This means that $(C, \mu, P_1, P_2)$ is in the second locus listed in (3.1) or in (3.2). In the remaining cases of a disconnecting node, if $\mu$ is the class of a stable $k$-differential whose aspects on the components of $C$ containing the two marked points are non-zero, then the two marked points are necessarily in the support of $\mu$, hence $(C, \mu, P_1, P_2)$ is in $\mathcal{Z}_2$. We are left with the case when one or both marked points are on a component of $C$ with zero $k$-differential. When only one marked point is on a component of $C$ with zero $k$-differential, we can find a twisted $k$-differential whose aspect on that component has a zero at the marked point and a pole of order at least $k + 1$ at the node. In order for such an element to be in $\pi_1^{-1}(\mathcal{Z}_1) \cap \pi_2^{-1}(\mathcal{Z}_1)$, the other marked point must be in the support of $\mu$. This puts $(C, \mu, P_1, P_2)$ in $\mathcal{Z}_2$. Now consider the case when both marked points are on a component of $C$ with zero $k$-differential. If the underlying component is of genus 1 and when $k = 1$, we end up with the third locus listed in (3.1). If the underlying component is of higher genus or if $k \geq 2$, then $(C, \mu, P_1, P_2)$ is the general element of a locus of codimension higher than two. The incidence variety compactification informs us that such an element is in the boundary of $\mathcal{Z}_2$: indeed, it is possible to find a twisted $k$-differential whose aspect on the component of $C$ with zero $k$-differential has a zero at the two marked points and a pole of order at least $k + 1$ at the node.

It remains to analyze the case of two-nodal curves. It suffices to show that for the pullback of every codimension two boundary component from $\overline{\mathcal{M}}_{g,2}$, its general element $(C, \mu, P_1, P_2)$ is not in $\pi_1^{-1}(\mathcal{Z}_1) \cap \pi_2^{-1}(\mathcal{Z}_1)$, with the exception of the loci of the fourth type listed in (3.1) when $k = 1$. If $C$ has only disconnecting nodes and nonrational components, then the points $P_1$ and $P_2$ are generically away from the support of $\mu$, hence such an element is not in $\pi_1^{-1}(\mathcal{Z}_1) \cap \pi_2^{-1}(\mathcal{Z}_1)$. Likewise, a general element of a locus of two-nodal curves with a twice marked rational tail is not in $\pi_1^{-1}(\mathcal{Z}_1) \cap \pi_2^{-1}(\mathcal{Z}_1)$, since the node at the attaching point of the rational tail is generically away from the support of $\mu$. If $C$ contains a rational bridge with one marked point, then the other marked point is generically away from the support of $\mu$. If instead $C$ has a twice marked rational bridge, then we have one of the loci of the fourth type listed in (3.1), and as discussed above, such loci are in $\pi_1^{-1}(\mathcal{Z}_1) \cap \pi_2^{-1}(\mathcal{Z}_1)$ but not in $\mathcal{Z}_2$ only when $k = 1$. Similar arguments cover the case when $C$ is irreducible with two non-disconnecting nodes, as well as the case when $C$ has both a non-disconnecting node and a disconnecting node. We are left with the case when $C$ has two components meeting in two points. If both components of $C$ are nonrational, the aspects of $\mu$ on both components have generically poles of order $k$ at both nodes, and the marked points are generically not in the support of such $k$-differentials. If $C$ has a (marked) rational component, the $k$-differential on the genus $g - 1$ component has generically poles of order $k$ at the two nodes, forcing there to be no zeros of the $k$-differential on the rational component. This shows that the loci in (3.1) and in (3.2) are the only components in the intersection $\pi_1^{-1}(\mathcal{Z}_1) \cap \pi_2^{-1}(\mathcal{Z}_1)$ when $k = 1$ and $k \geq 2$, respectively.

$(ii)$ Part $(i)$ implies

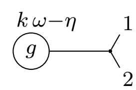
$$\pi_1^*[\mathcal{Z}_1]\cdot\pi_2^*[\mathcal{Z}_1] = [\mathcal{Z}_2] + a$$  $$+\, b$$  $$+\sum_{i=1}^{\lfloor g/2\rfloor} c_i$$  $$\in A^2\left(\mathbb{P}\left(\mathbb{E}^1_{g,2}\right)\right),$$

and $$\pi_1^*[\mathcal{Z}_1]\cdot\pi_2^*[\mathcal{Z}_1] = [\mathcal{Z}_2] + a$$  $$\in A^2\left(\mathbb{P}\left(\mathbb{E}^k_{g,2}\right)\right), \qquad \text{for } k\geq 2,$$

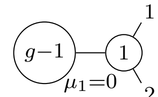
(3.3)

for some $a, b, c_i \in \mathbb{Q}$, with $i = 1,\ldots,\lfloor g/2\rfloor$. To find the coefficient $a$, let $\pi : \mathbb{P}\left(\mathbb{E}^k_{g,2}\right) \to \mathbb{P}\left(\mathbb{E}^k_{g}\right)$ be the forgetful map and consider $\pi_*$ of (3.3). Since

$$\pi_*\left(\pi_1^*[\mathcal{Z}_1]\cdot\pi_2^*[\mathcal{Z}_1]\right) = k^2(2g-2)^2\left[\mathbb{P}\left(\mathbb{E}^k_g\right)\right],$$

$$\pi_*\left(\mathcal{Z}_2\right) = k(2g-2)(k(2g-2)-1)\left[\mathbb{P}\left(\mathbb{E}^k_g\right)\right], \qquad \pi_*\Bigg($$  $$\Bigg) = k(2g-2)\left[\mathbb{P}\left(\mathbb{E}^k_g\right)\right],$$

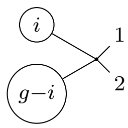
for all $k \geq 1$ and the push-forward of the other classes in (3.3) vanishes, one concludes that $a = 1$. This concludes the proof of the statement for $k \geq 2$. For $k = 1$, to find the coefficient $b$, we restrict to a test surface $S$ defined as follows: consider a pencil of plane cubics along with a section of the Hodge bundle over $\overline{\mathcal{M}}_{1,1}$, identify one of its basepoints with a general point on a fixed general genus $g-1$ curve with a general abelian differential, vary the first marked point along the genus $g-1$ component, and let the second marked point be one of the other basepoints of the pencil of plane cubics. The test surface $S$ has the following intersection profile

$$S\cdot\pi_1^*[\mathcal{Z}_1]\cdot\pi_2^*[\mathcal{Z}_1] = 2(g-1)-1, \qquad S\cdot\mathcal{Z}_2 = 2(g-1)-2, \qquad S\cdot$$  $$= 1,$$

and zero intersection with the remaining classes. The intersection with $\mathcal{Z}_2$ and $\pi_1^*[\mathcal{Z}_1]\cdot\pi_2^*[\mathcal{Z}_1]$ can be computed using the description of the boundaries of these loci provided by the incidence variety compactification. The section of the Hodge bundle over $\overline{\mathcal{M}}_{1,1}$ assigns the zero abelian differential to precisely one elliptic curve in the pencil of plane cubics. The test surface $S$ intersects $\mathcal{Z}_2$ transversely along the elements with such an elliptic tail, and with the marked point on the genus $g-1$ component coinciding with one of the zeros of the differential. This is because we may always find a twisted differential whose aspect on the elliptic tail has a zero at the marked point and a pole of order 2 at the node. The intersection with $\pi_1^*[\mathcal{Z}_1]\cdot\pi_2^*[\mathcal{Z}_1]$ consists of the $2(g-1)-2$ contributions just described, and one additional contribution given by the element of $S$ with the same elliptic tail with vanishing differential, obtained when the marked point on the genus $g-1$ component collides with the node. Such an element is clearly in $\pi_1^{-1}(\mathcal{Z}_1)$, because the second marked point is in the elliptic tail with a zero differential, so that a suitable twisted differential can be found as above, and it is also seen to be in $\pi_2^{-1}(\mathcal{Z}_1)$ by taking a twisted differential whose aspect on the rational bridge has poles of order 2 at both nodes and simple zeros at 2 smooth points. Such a twisted differential satisfies the global residue condition of the incidence variety compactification. (This element of $S$ is not in $\mathcal{Z}_2$: since the aspect of the twisted differential on the rational bridge has a pole of order 2 at the nodes, the aspect on the elliptic tail is necessarily non-zero, hence does not vanish at the second marked point.) Thus, we conclude that $b=1$.

To find the coefficients $c_i$, consider the test surface $S_i$, for $1\leq i\leq \lfloor g/2\rfloor$, obtained by taking a general element of the boundary divisor $\Delta_{i,\{1\}}$ in $\mathbb{P}\left(\mathbb{E}^1_{g,2}\right)$ and varying the two marked points on their corresponding components. This test surface has the following intersection profile

$$S_i \cdot \pi_1^*[\mathcal{Z}_1] \cdot \pi_2^*[\mathcal{Z}_1] = (2i-1)(2(g-i)-1), \quad S_i \cdot \mathcal{Z}_2 = (2i-1)(2(g-i)-1) - 1,$$

$S_i \cdot$  $= 1,$

and zero intersection with the remaining classes. The test surface intersects $\mathcal{Z}_2$ either when both marked points coincide with zeros of the differentials, or when exactly one of the marked points collides with the node and the other marked point is a zero of the differential. This is because, as before, we can find a twisted differential on the rational bridge that has a zero at the marked point, poles of order 2 at the nodes, and satisfies the global residue condition. When both marked points collide with the node, one can see that we obtain an additional contribution to the intersection with $\pi_1^*[\mathcal{Z}_1] \cdot \pi_2^*[\mathcal{Z}_1]$, since there exists a twisted differential vanishing at one, or at the other marked point. However, such an element is not in $\mathcal{Z}_2$, since there exist no twisted differential vanishing at *both* marked points. Indeed, in order for both of the marked points to be zeros of aspects $\eta_1$ and $\eta_2$ of a twisted differential on the two rational bridges, $\eta_1$ and $\eta_2$ must each have a pole of order $\geq 1$ at the node $Q$ where the two rational components meet. Since the sum of the order of the poles at $Q$ has to be equal to 2, one also has that $\eta_1$ and $\eta_2$ must each have a simple pole at $Q$. The global residue condition requires that $\operatorname{Res}_{Q_1} \eta_1 = \operatorname{Res}_{Q_2} \eta_2 = 0$, where $Q_1$ and $Q_2$ denote the nodes where the rational components meet the genus $i$ and $g-i$ components, respectively. By the residue theorem, this forces the residues at $Q$ to be zero, a contradiction. It follows that $c_i = 1$, for all $1 \leq i \leq \lfloor g/2 \rfloor$, hence the statement. □

Combining Proposition 3.2 and Lemma 3.1 yields Theorem 1.1.

## 3.2 A recursive formula for $\mathcal{Z}_n$

We will now prove the recursive formula for $\mathcal{Z}_n$, Theorem 1.2.

*Proof of Theorem 1.2.* Let $\pi_n^*[\mathcal{Z}_{n-1}] \cdot \rho_n^*[\mathcal{Z}_1] = [\mathcal{Z}_n] + \mathsf{E}_n$ and denote by $L_n$ and $R_n$ the left and right hand sides of this equation. We will use an inductive strategy to determine $\mathsf{E}_n$. As a base case, note that when $n = 2$, the proposition holds by Theorem 1.1. For the inductive step, assume that

$$\pi_{n-1}^*[\mathcal{Z}_{n-2}] \cdot \rho_{n-1}^*[\mathcal{Z}_1] = [\mathcal{Z}_{n-1}] + \sum_{\boldsymbol{I}} |\boldsymbol{I}|[E_{\boldsymbol{I}}] \in A^{n-1}\left(\mathbb{P}\left(\mathbb{E}_{g,n-1}^k\right)\right)$$

where the sum in the above expression is taken over all $\boldsymbol{I} \subseteq \{P_1, \ldots, P_{n-2}\}$ and the rational tail on $E_{\boldsymbol{I}}$ above contains the points specified by $\boldsymbol{I}$ along with $P_{n-1}$.

**Claim 1.** *We have*

$$\begin{aligned}(\pi_i)_*[L_n] &= [k(2g-2)-(n-2)]L_{n-1}\\ (\pi_i)_*[\mathsf{E}_n] &= [\mathcal{Z}_{n-1}] + [k(2g-2)-(n-2)]\mathsf{E}_{n-1}.\end{aligned}$$

*Proof of Claim 1.* First note that $\rho_n = \pi_1 \circ \cdots \circ \pi_{n-1}$. Since we may forget the points in any order, we also have that $\rho_n = \pi_1 \circ \cdots \circ \pi_{i-1} \circ \pi_{i+1} \circ \cdots \circ \pi_{n-1} \circ \pi_i$, where $\pi_j$ for $i+1 \leq j \leq n-1$ is the map which forgets the original $j$th marked point, before renumbering. One has

$$\begin{aligned}(\pi_i)_*[L_n] &= (\pi_i)_*\left(\pi_n^*[\mathcal{Z}_{n-1}]\cdot\rho_n^*[\mathcal{Z}_1]\right)\\ &= (\pi_i)_*\left(\pi_n^*[\mathcal{Z}_{n-1}]\right)\cdot\left(\pi_{n-2}^*\circ\cdots\circ\pi_1^*\right)[\mathcal{Z}_1]\\ &= (\pi_i)_*\left(\pi_n^*[\mathcal{Z}_{n-1}]\right)\cdot\left(\rho_{n-1}^*\right)[\mathcal{Z}_1]\\ &= [k(2g-2)-(n-2)]\left(\pi_{n-1}^*[\mathcal{Z}_{n-2}]\cdot\rho_{n-1}^*[\mathcal{Z}_1]\right)\\ &= [k(2g-2)-(n-2)]L_{n-1}\end{aligned}$$

where the coefficient $k(2g-2)-(n-2)$ comes from the number of potential locations for the $i$th marked zero. Note also that in the second line we renumber the points after applying $(\pi_i)_*$ so that each $P_j$ for $j>i$ is relabelled $P_{j-1}$.

Now let $a := k(2g-2)-(n-2)$ and note that $(\pi_i)_*[\mathcal{Z}_n] = (a-1)[\mathcal{Z}_{n-1}]$. Then,

$$aL_{n-1} = (\pi_i)_*[L_n] = (\pi_i)_*(\mathcal{Z}_n + \mathsf{E}_n) = (a-1)[\mathcal{Z}_i] + (\pi_i)_*(\mathsf{E}_n).$$

Thus, by induction, we must have that $(\pi_i)_*(\mathsf{E}_n) = [\mathcal{Z}_{n-1}] + a\mathsf{E}_{n-1}$, where

$$\mathsf{E}_{n-1} = \sum_{\boldsymbol{I}} |\boldsymbol{I}|[E_{\boldsymbol{I}}] \in A^{n-1}\left(\mathbb{P}\left(\mathbb{E}^k_{g,n-1}\right)\right).$$

△

**Claim 2.** *The $E_{\boldsymbol{I}}$ are in the support of $\mathsf{E}_n$.*

*Proof of Claim 2.* To see that $[E_{\boldsymbol{I}}]$ is in $\pi_n^*[\mathcal{Z}_{n-1}]$, we need to construct twisted $k$-differentials on both components that satisfy the conditions of the incidence variety compactification. The twisted differential on the genus $g$ component will be the nonzero stable $k$-differential on that curve, while the twisted differential on the rational component will have a pole of order $-2k - |\boldsymbol{I}|$ at the node and zeros at the $|\boldsymbol{I}|$ marked points on that component. When the $k$-differential on the genus $g$ component is not the $k$-th power of an abelian differential, the global $k$-residue condition is automatically satisfied. On the other hand, the locus inside $E_{\boldsymbol{I}}$ consisting of elements with $k$th powers of abelian differentials on the genus $g$ component is of codimension $\geq 1$, as a result of our definition of $E_{\boldsymbol{I}}$.

In order for $[E_{\boldsymbol{I}}]$ to be in $\rho_n^*[\mathcal{Z}_1]$ we need that the genus $g$ component has a $k$-differential which has a zero at the node; since we have a zero of order $|\boldsymbol{I}|$ at the node this is satisfied. △

**Claim 3.** *Over $\mathcal{M}_{g,n}^{\mathsf{rt}}$, the support of $\mathsf{E}_n$ consists of only the $E_{\boldsymbol{I}}$.*

*Proof of Claim 3.* By Claim 2, we know that $\mathsf{E}_n = [B_n] + \sum_{\boldsymbol{I}} a_{\boldsymbol{I}}^n [E_{\boldsymbol{I}}]$ for some possibly nonzero class $[B_n]$. Let $B$ be an irreducible component in $B_n$ in rational tails. First assume that $(\pi_i)_*[B]$ is nonzero for some $i$ (where we relabel the points by shifting $P_{j+1}$ to $P_j$ where $j \geq i$). By induction, this means that $(\pi_i)_*[B]$ is either $[\mathcal{Z}_{n-1}]$ or some $[E_{\boldsymbol{I}}]$ at the $(n-1)$th step. If the pushforward is $[\mathcal{Z}_{n-1}]$, then $B$ must be a component where general elements have a rational tail with points $\{P_i, P_j\}$ for some $j$. However, since $B$ must also be in $\pi_n^*[\mathcal{Z}_{n-1}]$ the only possibility is that $B$ is $E_{\boldsymbol{I}}$ where $\boldsymbol{I} = \{P_i\}$.

On the other hand, if the pushforward of $B$ is some $[E_{\boldsymbol{I}}]$ at the $(n-1)$th step, we will show that the $i$th marked point must have been on the genus $g$ component, which implies that $B$ itself is one of the $E_{\boldsymbol{I}}$ at the $n$th step. We will enumerate the possibilities otherwise and show that none can be true. If $B$ is a one-nodal locus with $\{P_i, P_n\}$ on a rational tail, then $(\pi_i)_*[B]$ is $[\mathcal{Z}_{n-1}]$ as previously discussed. If the $i$th marked point is with $\geq 2$ marked points on the rational tail, then the pushforward will vanish.

Now consider the case where a general element of $B$ has two nodes. If $B$ has more than one node, we will show that such a locus will have codimension $> n$. Consider the possible two-nodal loci where general elements have (a) two rational tails each attached to the genus $g$ component with the $i$th marked point on a rational tail with only one other marked point, (b) the $i$th marked point alone on a rational bridge, and (c) the $i$th marked point together with another point on a rational tail attached to a rational bridge. We will show that the codimension of the first locus is $n+1$ and leave to the reader the remaining cases for which a similar argument is valid. Such a locus $B$ as described in (a) has $|\boldsymbol{I}|+1$ points on one rational tail, 2 marked points on the other, and $n-|\boldsymbol{I}|-3$ marked points on the genus $g$ component. Meanwhile, the $k$-differential on the genus $g$ component must have zeros of multiplicities $|\boldsymbol{I}|$ and 2 at the corresponding nodal points, respectively. These zero multiplicities ensure that $(\pi_i)_*[B]$ is some $[E_{\boldsymbol{I}}]$ and that $B$ is in $\pi_n^*[\mathcal{Z}_{n-1}]$. Since this underlying two-nodal locus has codimension two in $\overline{\mathcal{M}}_{g,n}$, the codimension of such a $B$ is $(n-|\boldsymbol{I}|-3)+|\boldsymbol{I}|+2+2 = n+1$.

This exhausts the possibilities for $B$ where $(\pi_i)_*[B]$ is some $[E_{\boldsymbol{I}}]$ and the $i$th marked point in $B$ is on a rational tail. Thus, the $i$th marked point must be on the genus $g$ component in $B$ and if $(\pi_i)_*[B] = [E_{\boldsymbol{I}}]$, then $B = E_{\boldsymbol{I}'}$ where if $P_j \in \boldsymbol{I}$ and $j > i$, then $P_{j+1} \in \boldsymbol{I}'$, as a result of point relabelling.

Now consider the case where all pushforwards $(\pi_i)_*[B]$ vanish. This means that all marked points must be on rational components. Otherwise, if the the $i$th marked point were on the genus $g$ component, applying $(\pi_i)_*$ would not kill $B$. One codimension $n$ locus both in rational tails and in $\pi_n^*\left[\mathcal{Z}_{n-1}\right] \cdot \rho_n^*\left[\mathcal{Z}_1\right]$ satisfying this condition is $E_{\{P_1,\ldots,P_{n-1}\}}$. A locus satisfying these conditions must be one-nodal because any locus with more nodes is a specialization of this one and will have higher codimension, by the argument above. Without any restrictions on $n$ for $k > 1$, we may also have a locus $F_{\boldsymbol{I}}$ in which all marked points are on the rational tail and the $k$-differential on the genus $g$ component has a zero of order $|\boldsymbol{I}| = n-1$ at the node and is a $k$th power of an abelian differential. Note that such a locus is in $\pi_n^* [\mathcal{Z}_{n-1}] \cdot \rho_n^* [\mathcal{Z}_1]$. However, when $k = 2$ we restrict to the case $n \leq k(2g-2)-2$. This means that $|\boldsymbol{I}| \leq 4g-7$. No stratum of the form $\overline{\mathbb{H}}_g^2(|\boldsymbol{I}|, 1^{4g-4-|\boldsymbol{I}|})$ where $|\boldsymbol{I}| \leq 4g-7$ has the same dimension as a stratum having a component parametrizing $k$th powers of abelian differentials (see Theorem 1.1 of [BCG+19]). In the case when $|\boldsymbol{I}| = 4g-6$, the stratum $\overline{\mathbb{H}}_g^2(4g-6, 1^2)$ has the same dimension as the stratum $\overline{\mathbb{H}}_g^2(4g-6, 2)$, and when $|\boldsymbol{I}| = 4g-5$, the stratum $\overline{\mathbb{H}}_g^2(4g-5, 1)$ has the same dimension as $\overline{\mathbb{H}}_g^2(4g-4)$. Similarly, when $k \geq 3$, only $\overline{\mathbb{H}}_g^k(k(2g-2)-1, 1)$ has the same dimension as a stratum parametrizing $k$-powers of abelian differentials $\overline{\mathbb{H}}_g^k(k(2g-2))$. Thus, due to our restrictions in statement ($i$), we will never see a component of the form $[F_{\boldsymbol{I}}]$ in $\pi_n^* [\mathcal{Z}_{n-1}] \cdot \rho_n^* [\mathcal{Z}_1]$. △

**Claim 4.** *When $n \leq k$, $\mathsf{E}_n$ consists of only loci of curves with rational tails.*

*Proof of Claim 4.* By Claim 3, we know that in the expression $\mathsf{E}_n = [B_n] + \sum_{\boldsymbol{I}} a_{\boldsymbol{I}}^n [E_{\boldsymbol{I}}]$, $B_n$ must be some locus outside of rational tails. We will consider possible loci $B \subseteq B_n$. All loci $B$ for which some $(\pi_i)_*[B]$ is nonzero are not in $B_n$ since by induction a nonzero $(\pi_i)_*[B]$ must be either $[\mathcal{Z}_{n-1}]$ or some $[E_{\boldsymbol{I}}]$, and for $B$ outside of rational tails $(\pi_i)_*[B]$ cannot be either. For each possible $B$ we will argue in this way or by showing that the codimension of the locus is $> n$.

Consider a locus $B \subset \mathbb{P}\left(\mathbb{E}_{g,n}^k\right)$ whose general element is a curve with one disconnecting node, a genus $i$ component, and a genus $g-i$ component, where $i \geq 1$. If the general element has nonzero stable differentials on both components and the $n$ marked points are all zeros of these stable differentials, then this locus is of codimension $n+1$ and is entirely contained in $\mathcal{Z}_n$. Note that the locus where curves have more than one node, all components have genera $> 0$, and all $k$-differentials on the components are nonzero, is of strictly higher codimension and is also contained in $\mathcal{Z}_n$.

Now consider instead the case where the stable differential on the genus $i$ component is identically zero, while the differential on the genus $g-i$ component has zeros at all marked points on that component. Note that if a point $P_i$ is on the component with zero $k$-differential, then $(\pi_i)_*[B]$ is 0 and so this locus could potentially be in $B_n$. The codimension of this locus is $N := h^0(K_{C_i}^{\otimes k}(kq))+1 = k(2i-1)-i+2$, where $C_i$ is the genus $i$ component and $q$ is the node. Imposing the condition $n \leq k$ gives us that $n < n(2i-1)-i+2 \leq N$. Thus, this locus has codimension higher than $n$ and therefore cannot appear in $B_n$. Moreover, the locus parametrizing curves with more than one disconnecting node, and where a $k$-differential on a component with genus $> 0$ is zero is of strictly larger codimension.

The case where the locus $B \subset \mathbb{P}\left(\mathbb{E}^k_{g,n}\right)$ has general elements having one non-disconnecting node and zeros at all marked points is also of codimension $n+1$ and is in $\mathcal{Z}_n$.

Next consider the locus parametrizing curves with two disconnecting nodes, a genus $i$ component, a genus $g-i$ component, a rational tail with three or more marked points, and nonzero $k$-differentials on the nonrational components. If a point $P_i$ is on the rational tail, then $(\pi_i)_*[B]$ is zero. If $P_n$ is on the rational component, this locus is a specialization of some $E_{\boldsymbol{I}}$ and so it must have higher codimension. If $P_n$ is not on the rational component, this locus is also of higher codimension and entirely contained in $\mathcal{Z}_n$. The locus where we instead have a rational bridge with two or more marked points and nonzero $k$-differentials on the nonrational components is also of higher codimension. If $P_n$ is not on the rational bridge, then in order for this locus to be in $\pi_n^*[\mathcal{Z}_{n-1}]$, we must have poles of order $\geq -k+1$ at the nodes on the nonrational components. This condition allows the marked points on the rational bridge to be zeros of the twisted differential on that component. For the case where the rational component has two marked points $P_i$ and $P_j$, we have

$$\dim \overline{\mathbb{H}}_i^k(-k+1, 1^{k(2i-2)+k-1}) + \dim \overline{\mathbb{H}}_{g-i}^k(-k+1, 1^{k(2(g-i)-2)+k-1}) = 2gk - 2k + 2g - 6.$$

After accounting for one degree of freedom from the marked points on the rational bridge and another degree of freedom from the relative scaling of the $k$-differentials on the two components, one has that the dimension of this locus is $2gk-2k+2g-4$, which is codimension $n+1$. The case when $P_n$ is on the rational bridge is, in fact, no different; we must again have poles of order $\geq -k+1$ at the nodes on the nonrational components since a pole of order $-k$ will put a component on the same level in the level graph as the rational bridge and this would contradict the nonzero $k$-differential on that component. Thus, this locus too, has codimension $n+1$.

Finally, consider the locus of curves with a genus $g-1$ component attached at two points to a rational component with two or more marked points. If $P_i$ is on the rational component, then $(\pi_i)_*[B]$ is 0. We can again consider two possible cases: $P_n$ is on the rational component or it is not, and follow the same strategy as detailed in the previous paragraph to argue that this locus has higher codimension. △

**Claim 5.** *The classes $[\mathcal{Z}_n]$ and the classes $[E_{\boldsymbol{I}}]$ for all $\boldsymbol{I}$ are independent.*

*Proof of Claim 5.* When $n=2$, there is only one $E_{\boldsymbol{I}}$ and we can use the intersection data in the proof of Proposition 3.2 to see that $[\mathcal{Z}_2]$ and $[E_{\boldsymbol{I}}]$ are independent; since $[\mathcal{Z}_2] = (k\omega_1 - \eta)(k\omega_2 - \eta) - E_{\boldsymbol{I}}$, we get that $\mathcal{Z}_2 \cdot S = k^2(2g-3)$ whereas $E_{\boldsymbol{I}} \cdot S = 0$.

We will proceed by induction on $n$. Let $n \geq 3$ and $\alpha_n[\mathcal{Z}_n] + \sum_{\boldsymbol{I}} \beta_{\boldsymbol{I}} E_{\boldsymbol{I}} = 0$ and suppose that no relation exists for smaller $n$. If we apply $(\pi_i)_*$ to this expression we get

$$\left((a-1)\alpha_n + \beta_{\{P_i\}}\right)[\mathcal{Z}_n] + a\sum_{\boldsymbol{I}} \beta_{\boldsymbol{I}} E_{\boldsymbol{I}}$$

where $a = k(2g-2)-(n-2)$. Since there is no relation for lower $n$ by assumption, we must have that $(a-1)\alpha_n + \beta_{\{P_i\}} = 0$ which implies $\alpha_n = \beta_{\{P_i\}}/(a-1)$. Similarly, all $\beta_{\boldsymbol{I}}$ for which $P_i \notin \boldsymbol{I}$ must be 0 since $(\pi_i)_*[E_{\boldsymbol{I}}]$ does not vanish and would otherwise yield a nontrivial relation for $n-1$ marked points. Applying $(\pi_i)_*$ for $1 \leq i \leq n-1$ shows that $\beta_{\boldsymbol{I}} = 0$ in all cases except possibly $\boldsymbol{I} = \{P_1, \dots, P_{n-1}\}$. Since $\alpha_n = \beta_{\{P_i\}}/(a-1)$, we also get that $\alpha_n = 0$. Thus, we are left with $\beta_{\{P_1,\dots,P_{n-1}\}} E_{\{P_1,\dots,P_{n-1}\}} = 0$. From the test locus computation in the last paragraph of the proof of Claim 6, we can see that the locus $E_{\{P_1,\dots,P_{n-1}\}}$ is not trivial and so $\beta_{\{P_1,\dots,P_{n-1}\}} = 0$ as well. △

By the above claims, we know that $L_n = z_n[\mathcal{Z}_n] + \sum_{\boldsymbol{I}} a^n_{\boldsymbol{I}}[E_{\boldsymbol{I}}]$ and all that is left is to compute the coefficients $z_n$ and $a^n_{\boldsymbol{I}}$.

**Claim 6.** *$z_n = 1$ and $a^n_{\boldsymbol{I}} = |\boldsymbol{I}|$.*

*Proof of Claim 6.* Applying $(\pi_i)_*$ to $L_n = z_n[\mathcal{Z}_n] + \sum_{\boldsymbol{I}} a^n_{\boldsymbol{I}}[E_{\boldsymbol{I}}]$ gives

$$aL_{n-1} = z_n(a-1)[\mathcal{Z}_{n-1}] + a^n_{\{P_i\}}[\mathcal{Z}_{n-1}] + \sum_{\boldsymbol{I}, P_i \notin \boldsymbol{I}} a a^n_{\boldsymbol{I}}[E_{\overline{\boldsymbol{I}}}]$$

where $\overline{\boldsymbol{I}}$ is obtained from $\boldsymbol{I}$ by shifting down by one all markings labelled $> i$. By induction, the expression above is equal to

$$aR_{n-1} = a[\mathcal{Z}_{n-1}] + \sum_{\boldsymbol{I}} a a^{n-1}_{\boldsymbol{I}}[E_{\boldsymbol{I}}]$$

where the $[E_{\boldsymbol{I}}]$ here are precisely the $[E_{\overline{\boldsymbol{I}}}]$ from the above summation; indeed, note that $P_{n-1} \in \overline{\boldsymbol{I}}$. By Claim 5, we may determine $z_n$ and $a_{\boldsymbol{I}}$ by simply comparing the coefficients of the appropriate expressions. We have relations $z_n(a-1) + a^n_{\{P_i\}} = a$ and $a^n_{\boldsymbol{I}} = a^{n-1}_{\overline{\boldsymbol{I}}}$. Using induction and instead applying $(\pi_j)_*$ to $L_n$ where $i \neq j$, gives that $a^n_{\{P_i\}} = 1$. This implies $z_n = 1$. Also by induction we have $a^n_{\boldsymbol{I}} = |\boldsymbol{I}|$. Since for all $[E_{\boldsymbol{I}}]$, except when $|\boldsymbol{I}| = n-1$, there exists some $i$ for which $(\pi_i)_*[E_{\boldsymbol{I}}]$ does not vanish, we have found all coefficients with the exception of $a_{\{P_1,\dots,P_{n-1}\}}$.

To find $a_{\{P_1,\dots,P_{n-1}\}}$, consider the $n$-dimensional test space $S$ consisting of a general genus $g$ curve with a stable differential chosen from a general $\mathbb{P}^{n-1}$, attached at a general point to an $(n-1)$-marked rational tail, and let the $n$th marked point vary along the genus $g$ component. This test locus has the following intersection data:

$$S \cdot E_{\{P_1,\dots,P_{n-1}\}} = 1 \qquad S \cdot E_{\boldsymbol{I}} = 0,\ |\boldsymbol{I}| < n-1$$
$$S \cdot \mathcal{Z}_n = a-1 \qquad S \cdot L_n = k(2g-2)$$

Note that the intersection $S \cdot L_n = (a-1) + (n-1)$ where the second term is a result of the $n$th marked point meeting the node, where the differential on the genus $g$ component has a zero of multiplicity $n-1$. This intersection data gives that $a_{\{P_1,\dots,P_{n-1}\}} = n-1$. △

□

# 4 Divisors of $k$-differentials supported at one Brill-Noether special point

After reviewing the required background on pointed Brill-Noether divisors in §4.1, we prove Theorems 1.3 and 1.4 in §4.2.

## 4.1 Background on pointed Brill-Noether divisors

For a smooth algebraic curve $C$, the variety $G^r_d(C)$ parametrizes linear series of degree $d$ and projective dimension $r$. For $\ell = (L, V)$ in $G^r_d(C)$, the *vanishing sequence* of $\ell$ at a point $P$ in $C$

$$\boldsymbol{a}^\ell(P) : 0 \leq a_0 < \cdots < a_r \leq d$$

is defined as the increasing sequence of vanishing orders of sections in $V$ at $P$. Given sequences $\boldsymbol{a}^i = (a^i_0, \ldots, a^i_r)$, for $i = 1, \ldots, n$, the *adjusted Brill-Noether number* is defined as

$$\rho\left(g, r, d, \boldsymbol{a}^1, \ldots, \boldsymbol{a}^n\right) := g - (r+1)(g - d + r) - \sum_{i=1}^{n} \sum_{j=0}^{r} \left(a^i_j - j\right).$$

The pointed version of the Brill-Noether Theorem [EH87] states that a general pointed curve $(C, P)$ of genus $g > 0$ admits a linear series $\ell \in G^r_d(C)$ with vanishing sequence $\boldsymbol{a}^\ell(P) = \boldsymbol{a}$ if and only if

$$\sum_{i=0}^{r} (a_i - i + g - d + r)_+ \leq g,$$

where $(n)_+ := \max\{n, 0\}$, for $n \in \mathbb{Z}$. This condition is stronger than $\rho(g, r, d, \boldsymbol{a}) \geq 0$.

When $g \in \{0, 1\}$, one has $\rho(g, r, d, \boldsymbol{a}^\ell(P)) \geq 0$ for any $\ell \in G^r_d(C)$ and any $P \in C$. However, when $g \geq 2$ and for $r, d, \boldsymbol{a}$ such that $\rho(g, r, d, \boldsymbol{a}) = -1$, the locus in $\mathcal{M}_{g,1}$ consisting of pointed curves $(C, P)$ admitting a linear series $\ell \in G^r_d(C)$ such that $\boldsymbol{a}^\ell(P) \geq \boldsymbol{a}$ is a proper subvariety with a divisorial component [EH89]. Thus, for $g \geq 2$ and $\boldsymbol{a} : 0 \leq a_0 < \cdots < a_r \leq d$ such that $\rho(g, r, d, \boldsymbol{a}) = -1$, one defines the *pointed Brill-Noether divisor* $\mathcal{M}^{\boldsymbol{a}}_{g,d}$ as the *divisorial* component of the locus in $\mathcal{M}_{g,1}$ consisting of pointed curves $(C, P)$ admitting a linear series $\ell \in G^r_d(C)$ with $\boldsymbol{a}^\ell(P) \geq \boldsymbol{a}$. For example, when $d = 2g - 2$, $r = g - 1$, and $\boldsymbol{a} = (0, 1, 2, \ldots, g - 2, g)$, then $\mathcal{M}^{\boldsymbol{a}}_{g,d}$ is the divisor of curves with a marked Weierstrass point.

The closure of $\mathcal{M}^{\boldsymbol{a}}_{g,d}$ in $\overline{\mathcal{M}}_{g,1}$ is denoted $\overline{\mathcal{M}}^{\boldsymbol{a}}_{g,d}$. These loci and their generalizations in [Log03] have been very useful: they appear in Logan's proof that $\overline{\mathcal{M}}_{g,n}$ is of general type for $4 \leq g \leq 23$ for large enough values of $n$ [Log03]; they were also used to establish the non-varying property of Lyapunov exponents for certain low genus strata of abelian and quadratic differentials [CM12, CM14].

After [EH89], the class of each $\overline{\mathcal{M}}^{\boldsymbol{a}}_{g,d}$ can be written as a linear combination

$$\left[\overline{\mathcal{M}}^{\boldsymbol{a}}_{g,d}\right] = \mu_{g,d,\boldsymbol{a}}\, \mathcal{BN}_g + \nu_{g,d,\boldsymbol{a}}\, \mathcal{W}_g \,\in\, \mathrm{Pic}\left(\overline{\mathcal{M}}_{g,1}\right), \qquad \text{for some } \mu_{g,d,\boldsymbol{a}}, \nu_{g,d,\boldsymbol{a}} \in \mathbb{Q}_{\geq 0}, \tag{4.1}$$

where $\mathcal{W}_g$ denotes the Weierstrass divisor class on $\overline{\mathcal{M}}_{g,1}$ equal to

$$\mathcal{W}_g := \binom{g+1}{2}\psi - \lambda - \sum_{i=1}^{g-1}\binom{g-i+1}{2}\delta_i \,\in\, \mathrm{Pic}\left(\overline{\mathcal{M}}_{g,1}\right), \tag{4.2}$$

and $\mathcal{BN}_g$ is the pullback of the Brill-Noether divisor class from $\overline{\mathcal{M}}_g$ which has class formula

$$\mathcal{BN}_g := (g+3)\lambda - \frac{g+1}{6}\delta_0 - \sum_{i=1}^{g-1} i(g-i)\delta_i \,\in\, \mathrm{Pic}\left(\overline{\mathcal{M}}_{g,1}\right). \tag{4.3}$$

Explicit formulae for $\mu_{g,d,\boldsymbol{a}}$ and $\nu_{g,d,\boldsymbol{a}}$ were computed in [FT16], and make an appearance in our computation. These values are expressed in terms of the number $n_{g,d,\boldsymbol{a}}$ of pairs $(P,\ell) \in C \times G^r_d(C)$ satisfying $\boldsymbol{a}^{\ell}(P) = \boldsymbol{a}$, where $C$ is a general curve of genus $g \geq 2$ and $\rho(g,r,d,\boldsymbol{a}) = -1$. Let $\delta^i_j$ be the Kronecker delta. After [FT16], one has

$$n_{g,d,\boldsymbol{a}} = g! \sum_{0\leq j_1<j_2\leq r} \left((a_{j_2}-a_{j_1})^2 - 1\right) \frac{\prod_{0\leq i<k\leq r}\left(a_k - \delta^{j_1}_k - \delta^{j_2}_k - a_i + \delta^{j_1}_i + \delta^{j_2}_i\right)}{\prod_{i=0}^{r}\left(g-d+r+a_i-\delta^{j_1}_i-\delta^{j_2}_i\right)!} \tag{4.4}$$

and

$$\begin{aligned}\mu_{g,d,\boldsymbol{a}} &= -\frac{n_{g,d,\boldsymbol{a}}}{2(g^2-1)} + \frac{1}{4\binom{g-1}{2}} \sum_{i=0}^{r} n_{g-1,d,(a_0+1-\delta_0^i,\ldots,a_r+1-\delta_r^i)}, \\ \nu_{g,d,\boldsymbol{a}} &= \frac{n_{g,d,\boldsymbol{a}}}{g(g^2-1)}.\end{aligned} \tag{4.5}$$

## 4.2 Proof of Theorems 1.3 and 1.4

We now prove Theorems 1.3 and 1.4 on the class of the divisors $\overline{\mathbb{W}}_g^k$ and $\overline{\mathbb{H}}_{g,d}^{\boldsymbol{a}}$. We proceed by considering the intersection of $\mathcal{Z}_1$ with the pullbacks of $\mathcal{W}_g$ and more generally $\overline{\mathcal{M}}_{g,d}^{\boldsymbol{a}}$ to $\mathbb{P}\left(\mathbb{E}_{g,1}^k\right)$, and then applying the push-forward via the map $\pi\colon \mathbb{P}\left(\mathbb{E}_{g,1}^k\right) \to \mathbb{P}\left(\mathbb{E}_g^k\right)$. In particular, we will prove the following lemma.

**Lemma 4.1.** *For any pointed Brill-Noether divisor $\overline{\mathcal{M}}_{g,d}^{\boldsymbol{a}} \subset \mathbb{P}\left(\mathbb{E}_{g,1}^k\right)$ pulled back from $\overline{\mathcal{M}}_{g,1}$, one has*

$$\left[\pi_*\left(\mathcal{Z}_1 \cdot \overline{\mathcal{M}}_{g,d}^{\boldsymbol{a}}\right)\right] = \left[\overline{\mathbb{H}}_{g,d}^{\boldsymbol{a}}\right] \qquad \in \operatorname{Pic}\left(\mathbb{P}\left(\mathbb{E}_g^k\right)\right).$$

Before proving Lemma 4.1, we show how it implies Theorems 1.3 and 1.4.

*Proof of Theorem 1.3.* Since $\mathcal{W}_g = \overline{\mathcal{M}}_{g,d}^{\boldsymbol{a}}$ in $\operatorname{Pic}\left(\overline{\mathcal{M}}_{g,1}\right)$ and $\overline{\mathbb{W}}_g^k = \overline{\mathbb{H}}_{g,d}^{\boldsymbol{a}}$ in $\operatorname{Pic}\left(\mathbb{P}\left(\mathbb{E}_g^k\right)\right)$ with $d = 2g-2$, $r = g-1$, and $\boldsymbol{a} = (0,1,2,\ldots,g-2,g)$, Lemma 4.1 implies $\overline{\mathbb{W}}_g^k = \pi_*\left(\mathcal{Z}_1 \cdot \mathcal{W}_g\right)$. Consider the intersection

$$\mathcal{Z}_1 \cdot \mathcal{W}_g = (k\psi_1 - \eta)\left(\binom{g+1}{2}\psi_1 - \lambda - \sum_{i=1}^{g-1}\binom{g-i+1}{2}\delta_i\right) \in A^2\left(\mathbb{P}\left(\mathbb{E}_{g,1}^k\right)\right),$$

where we used Lemma 3.1 and (4.2). The push-forward via the map $\pi\colon \mathbb{P}\left(\mathbb{E}_{g,1}^k\right) \to$

$\mathbb{P}\left(\mathbb{E}_g^k\right)$ is computed as

$$\pi_*\left(\mathcal{Z}_1\cdot\mathcal{W}_g\right) = -\,g(g^2-1)\eta + k\binom{g+1}{2}\kappa_1 - k(2g-2)\lambda$$
$$-\sum_{i=1}^{\lfloor g/2\rfloor} k\left((2i-1)\binom{g-i+1}{2} + (2g-2i-1)\binom{i+1}{2}\right)\delta_i \,\in\, \mathrm{Pic}\left(\mathbb{P}\left(\mathbb{E}_g^k\right)\right).$$

Here we used

$$\kappa_1 := \pi_*\left(\psi_1^2\right), \qquad \pi_*\left(\psi_1\lambda\right) = (2g-2)\lambda, \qquad \pi_*\left(\psi_1\eta\right) = (2g-2)\eta,$$
$$\pi_*\left(\psi_1\delta_i\right) = (2i-1)\delta_i \quad \text{and} \quad \pi_*\left(\psi_1\delta_{g-i}\right) = (2g-2i-1)\delta_i, \quad \text{for } 1\le i\le \lfloor g/2\rfloor.$$

Mumford's formula $\kappa_1 = 12\lambda - \sum_{i=0}^{\lfloor g/2\rfloor}\delta_i$ and simplifying yield

$$\pi_*\left(\mathcal{Z}_1\cdot\mathcal{W}_g\right) = -g(g^2-1)\eta + k(6g^2+4g+2)\lambda - k\binom{g+1}{2}\delta_0 - \sum_{i=1}^{\lfloor g/2\rfloor} k(g+3)i(g-i)\delta_i.$$

The statement follows. $\square$

*Proof of Theorem 1.4.* From (4.1), one has

$$\begin{aligned}\pi_*\left(\mathcal{Z}_1\cdot\overline{\mathcal{M}}^{\boldsymbol{a}}_{g,d}\right) &= \pi_*\left(\mathcal{Z}_1\cdot\left(\mu_{d,g,\boldsymbol{a}}\,\mathcal{BN}_g + \nu_{d,g,\boldsymbol{a}}\,\mathcal{W}_g\right)\right)\\ &= \pi_*\left(\left(k\psi_1-\eta\right)\left(\mu_{d,g,\boldsymbol{a}}\,\mathcal{BN}_g + \nu_{d,g,\boldsymbol{a}}\,\mathcal{W}_g\right)\right)\\ &= k(2g-2)\mu_{d,g,\boldsymbol{a}}\,\mathcal{BN}_g + \nu_{d,g,\boldsymbol{a}}\left[\pi_*\left(\mathcal{Z}_1\cdot\mathcal{W}_g\right)\right].\end{aligned}$$

The statement follows from Lemma 4.1. $\square$

Explicitly, the class of $\overline{\mathbb{H}}^{\boldsymbol{a}}_{g,d}$ is given by:

**Corollary 4.2.** *One has* $\left[\overline{\mathbb{H}}^{\boldsymbol{a}}_{g,d}\right] = c_\eta \eta + c_\lambda \lambda - \sum_{i=0}^{\lfloor g/2 \rfloor} c_i \delta_i \in \operatorname{Pic}\left(\mathbb{P}\left(\mathbb{E}^k_g\right)\right)$, *where*

$$c_\eta = -g(g^2-1)\nu_{g,d,\boldsymbol{a}}, \qquad c_\lambda = 2(g-1)(g+3)k\mu_{g,d,\boldsymbol{a}} + 2(3g^2+2g+1)k\nu_{g,d,\boldsymbol{a}},$$
$$c_0 = \frac{g^2-1}{3}k\mu_{g,d,\boldsymbol{a}} + \frac{g(g+1)}{2}k\nu_{g,d,\boldsymbol{a}}, \quad c_i = 2i(g-i)(g-1)k\mu_{g,d,\boldsymbol{a}} + i(g-i)(g+3)k\nu_{g,d,\boldsymbol{a}}, \ (i \geq 1),$$

*and* $\mu_{g,d,\boldsymbol{a}}$ *and* $\nu_{g,d,\boldsymbol{a}}$ *are given by* (4.5).

It remains to prove Lemma 4.1. We will need the following additional lemma.

**Lemma 4.3.** *Any nonzero effective divisor class of the form* $E := \sum_{i=0}^{\lfloor g/2 \rfloor} c_i \delta_i$ *on* $\overline{\mathcal{M}}_g$ *or on* $\mathbb{P}\left(\mathbb{E}^k_g\right)$ *for* $g \geq 3$ *and* $k \geq 1$ *is rigid.*

*Proof.* First, note that any such effective cycle class $E$ necessarily has $c_i \geq 0$ for all $i$. Moreover, after replacing $E$ with a positive multiple, we may assume that the $c_i$ are non-negative integers. Indeed, the rigidity of some positive multiple of $E$ implies the rigidity of $E$.

Suppose that there exists some effective cycle class $D$ such that $D \equiv mE$, for some $m > 0$, and that $D$ is not supported on the support of $E$. First consider the case when $E$ is a cycle class on $\overline{\mathcal{M}}_g$. For all boundary divisors $\delta_i$ with $i > 0$ such that $c_i > 0$, we can construct a moving curve $X_i$ in $\delta_i$ by attaching a general genus $i$ curve to a general genus $g-i$ curve at a point which moves along the genus $g-i$ component. We have $X_i \cdot \delta_i = 2 - 2(g-i)$ and $X \cdot \delta_j = 0$ for $j \neq i$. Thus since $D \cdot X_i < 0$ and $X_i$ is moving in $\delta_i$, we know that $D$ must be supported on $\delta_i$. Indeed, since $(D - (mc_i - 1)\delta_i) \cdot X_i < 0$ and $(D - mc_i\delta_i) \cdot X_i = 0$, we know that $D - mc_i\delta_i$ is effective. We proceed in this manner for all $i > 0$. This shows that $D - m\sum_{i=1}^{\lfloor g/2 \rfloor} c_i\delta_i$ is effective. Now consider the moving curve $X_0$ in $\delta_0$ formed by taking a genus $g-1$ curve and identifying a fixed point to a point which moves along the curve. We have $X_0 \cdot \delta_0 = 2 - 2g < 0$, which shows that $\delta_0$ is rigid. Since $D - m\sum_{i=1}^{\lfloor g/2 \rfloor} c_i\delta_i \equiv mc_0\delta_0$

and $\delta_0$ is rigid, we conclude that $D$ must also be supported on $\delta_0$. This contradicts our assumption that the support of $D$ does not contain the support of $E$.

Now assume $E$ is a cycle class on $\mathbb{P}\left(\mathbb{E}_g^k\right)$. For all $0 \leq i \leq \lfloor g/2 \rfloor$, we construct a moving curve in $\delta_i \subset \mathbb{P}\left(\mathbb{E}_g^k\right)$ from the previously defined $X_i$. Let $f\colon \mathbb{P}\left(\mathbb{E}_g^k\right) \to \overline{\mathcal{M}}_g$ and consider the intersection of $2k(g-1) - g - 1 + \delta_k^1$ general hyperplanes in $f^{-1}(X_i)$. When the intersecting hyperplanes are general, the resulting curve $X_i'$ is irreducible by Bertini's theorem. Moreover, by choosing general hyperplanes we avoid the possibility that $f$ contracts $X_i'$ to a point, so $X_i'$ must in fact cover $X_i$. Varying the intersecting hyperplanes does not change the numerical class of the resulting curve, so this construction produces a moving curve in $\delta_i \subset \mathbb{P}\left(\mathbb{E}_g^k\right)$. Since $f^*\delta_i \cdot X_i' = \delta_i \cdot f_* X_i' < 0$, the proof of the rigidity of $E$ follows as in the previous paragraph. □

*Proof of Lemma 4.1 when $g \geq 3$.* By definition the classes $\pi_*(\mathcal{Z}_1 \cdot \overline{\mathcal{M}}_{g,d}^{\boldsymbol{a}})$ and $\overline{\mathbb{H}}_{g,d}^{\boldsymbol{a}}$ agree on the interior of the moduli space $\mathbb{P}\left(\mathbb{E}_g^k\right)$. Indeed, one may verify the coefficients $c_\eta$ and $c_\lambda$ (as well as $c_0$) directly by using the first three test curves provided in §4.3. Note that here we deal exclusively with the case $g \geq 3$ and leave the $g = 2$ case for the following section.

We now want to rule out the possibility that $\pi_*(\mathcal{Z}_1 \cdot \overline{\mathcal{M}}_{g,d}^{\boldsymbol{a}}) = \overline{\mathbb{H}}_{g,d}^{\boldsymbol{a}} + E$, where $E$ is an effective cycle class of the form $E = \sum_{i=0}^{\lfloor g/2 \rfloor} c_i \delta_i$. Such an $E$ is rigid by Lemma 4.3 and so must be supported on boundary divisors $\delta_0, \ldots, \delta_{\lfloor g/2 \rfloor}$. Thus, it is enough to argue that the preimage under $\pi$ of a *general* element of $\delta_i$, for $0 \leq i \leq \lfloor g/2 \rfloor$, is disjoint from $\mathcal{Z}_1 \cap \overline{\mathcal{M}}_{g,d}^{\boldsymbol{a}}$. First consider the case $i > 0$. A general element of $\delta_i$ consists of a pair $(X, \mu)$, where the nodal curve $X$ is obtained by identifying the marked points of two *general* pointed curves $(C_1, Q_1)$ and $(C_2, Q_2)$ of genus $i$ and genus $g - i$, respectively, and $\mu$ is a *general* stable $k$-differential on $X$. Since there are finitely many points on $X$ which are limits of a Brill-Noether special point $P$ on a nearby smooth curve $C$ such that $\boldsymbol{a}^\ell(P) \geq \boldsymbol{a}$ for some $\ell \in G_d^r(C)$, the zeros of $\mu$ must avoid these Brill-Noether special points. Indeed, a general stable $k$-differential on $X$ consists of non-zero sections of $H^0(\omega_{C_1}^k(kQ_1))$ and $H^0(\omega_{C_2}^k(kQ_2))$ satisfying the $k$-residue condition. Moreover, general stable $k$-differentials avoid the finite collection of hyperplanes in $\mathbb{P}H^0(\omega_{C_1}^k(kQ_1))$ and $\mathbb{P}H^0(\omega_{C_2}^k(kQ_2))$ that parametrize $k$-differentials containing the Brill-Noether special points. The resulting hypersurface cut out by the residue condition in $H^0(\omega_{C_1}^k(kQ_1)) \times H^0(\omega_{C_2}^k(kQ_2))$ does not contain any of the Brill-Noether hyperplanes since for every Brill-Noether special point, one may find differentials on the two components such that one has a zero at the Brill-Noether special point but does not have a nodal $k$-residue compatible with the other. The case $i = 0$ is similar. A general element of $\delta_0 \subset \mathbb{P}\left(\mathbb{E}_g^k\right)$ has its stable $k$-differential having poles of order $k$ at the two nodal points in the normalized genus $g-1$ curve. The zeros of such a $k$-differential avoid the Brill-Noether special points by the same argument as before.

We must also verify that curves with a marked rational bridge above $\delta_i$ in $\mathbb{P}\left(\mathbb{E}_{g,1}^k\right)$ are not contained in $\mathcal{Z}_1 \cap \overline{\mathcal{M}}_{g,d}^{\boldsymbol{a}}$. However, since the adjusted Brill-Noether number $\rho(R, P, Q_1, Q_2)$ on a rational component $R$ meeting the genus $i$ and genus $g-i$ components at points $Q_1$ and $Q_2$, respectively, and having marked point $P$, is always nonnegative, we must have an adjusted Brill-Noether number equal to $-1$ on one of the other components. But this will not be true for a general element in $\delta_i$. Two-nodal curves lying above a general element of $\delta_0$ having a once-marked rational component will have aspects on both components that have poles of order $-k$ at the nodes. Since the twisted differential on the rational component has no zeros, such a curve will not lie in $\mathcal{Z}_1$. $\square$

## 4.3 Test families

In order to extend the proof of Lemma 4.1 to the case $g = 2$, we will use a few test curves. For fixed $g, r, d$ and a vanishing sequence $\boldsymbol{a} : 0 \le a_0 < \cdots < a_r \le d$ such that $\rho(g, r, d, \boldsymbol{a}) = -1$, write

$$\left[\overline{\mathbb{H}}^{\boldsymbol{a}}_{g,d}\right] = c_\eta\,\eta + c_\lambda\,\lambda - \sum_{i=0}^{\lfloor g/2 \rfloor} c_i\,\delta_i \in \mathrm{Pic}_{\mathbb{Q}}\left(\mathbb{P}\left(\mathbb{E}^k_g\right)\right), \qquad \text{for some } c_\eta, c_\lambda, c_i \in \mathbb{Q}.$$

### 4.3.1 The coefficient $c_\eta$

Let $C$ be a general genus $g$ curve $k$-canonically embedded in $\mathbb{P}^N$, where $N = g - 1$ for $k = 1$, and $N = (g-1)(2k-1) - 1$ for $k \ge 2$. Let $\Lambda \cong \mathbb{P}^{N-3}$ be a fixed general subspace and consider the one-dimensional family $A$ of hyperplanes in $\mathbb{P}^N$ containing $\Lambda$. Then $A \cdot \eta = -1$, while $A \cdot \lambda = 0$ and $A \cdot \delta_i = 0$ for all $i$. Moreover, $A \cdot \left[\overline{\mathbb{H}}^{\boldsymbol{a}}_{g,d}\right] = n_{g,d,\boldsymbol{a}}$ where $n_{g,d,\boldsymbol{a}}$ is the number in (4.4) of pairs $(P, \ell) \in C \times G^r_d(C)$ such that $\boldsymbol{a}^\ell(P) = \boldsymbol{a}$ [FT16]. Combining with (4.5), this gives

$$c_\eta = -g(g^2-1)\nu_{g,d,\boldsymbol{a}}.$$

### 4.3.2 Curves on K3 surfaces

We consider here a Lefschetz pencil of curves of genus $g \ge 3$ lying on a general K3 surface $S$ of degree $2g - 2$ in $\mathbb{P}^g$. Let $\mathcal{X}$ be the blow-up of $S$ at the $2g - 2$ base points of the pencil, and let $p \colon \mathcal{X} \to \mathbb{P}^1$ be the corresponding family of curves. Fix general genus $g$ curves $C_1, \ldots, C_k$ in $S$. The cycle $[C_1] + \cdots + [C_k]$ restricts to a $k$-canonical divisor on each fiber of $p$, hence this gives rise to a pencil $\tau \colon \mathbb{P}^1 \to \mathbb{P}\left(\mathbb{E}^k_g\right)$ in the projectivized $k$-Hodge bundle. The intersections with the generators are

$$\tau^*\eta = k, \qquad \tau^*\lambda = g+1, \qquad \tau^*\delta_0 = 6g+18, \qquad \tau^*\delta_i = 0, \qquad \text{for } i > 0.$$

The intersections with $\lambda$ and the $\delta_i$ are classical [CU93, FP05]. The degree of $\eta$ can be computed by intersecting the relation $\omega^k_{\mathcal{X}/\mathbb{P}^1} = p^*\eta \otimes \mathcal{O}_{\mathcal{X}}(C_1 + \cdots + C_k)$ valid on $\mathcal{X}$ with the class of one of the $2g-2$ exceptional divisors (see [Ghe21, Ex. 3.2] for more details).

To compute the intersection of the pencil $\tau$ with the class of a Brill-Noether divisor $\overline{\mathbb{H}}^{\boldsymbol{a}}_{g,d}$, we start from the locus of corresponding Brill-Noether special points in $\mathcal{X}$. The pull-back of the divisor class $\left[\overline{\mathcal{M}}^{\boldsymbol{a}}_{g,d}\right]$ from (4.1) via the moduli map $\mathcal{X} \to \overline{\mathcal{M}}_{g,1}$ is

$$\left[\mathcal{X}^{\boldsymbol{a}}_{g,d}\right] := \nu_{g,d,\boldsymbol{a}} \frac{g(g+1)}{2} c_1\left(\omega_{\mathcal{X}/\mathbb{P}^1}\right) + \left(\mu_{g,d,\boldsymbol{a}}(g+3) - \nu_{g,d,\boldsymbol{a}}\right) p^*\lambda - \mu_{g,d,\boldsymbol{a}} \frac{g+1}{6} p^*\delta_0. \quad (4.6)$$

The intersection of the pencil $\tau\colon \mathbb{P}^1 \to \mathbb{P}\left(\mathbb{E}^k_g\right)$ with a Brill-Noether divisor class $\left[\overline{\mathbb{H}}^{\boldsymbol{a}}_{g,d}\right]$ on $\mathbb{P}\left(\mathbb{E}^k_g\right)$ equals the intersection of $\left[\mathcal{X}^{\boldsymbol{a}}_{g,d}\right]$ with $p^*\left([C_1] + \cdots + [C_k]\right) = k\Big(f + \sum_{i=1}^{2g-2} E_i\Big)$. Here $f$ is the class of a fiber of $p$, and $E_i$ are the classes of the exceptional divisors. This gives

$$\tau^*\left[\overline{\mathbb{H}}^{\boldsymbol{a}}_{g,d}\right] = \left[\mathcal{X}^{\boldsymbol{a}}_{g,d}\right] \cdot k\left(f + \sum_{i=1}^{2g-2} E_i\right) = 2k(g+1)(g-1)^2\, \nu_{g,d,\boldsymbol{a}}.$$

Here we used that the nonzero intersections are given by $c_1\left(\omega_{\mathcal{X}/\mathbb{P}^1}\right) \cdot f = 2g-2$, $c_1\left(\omega_{\mathcal{X}/\mathbb{P}^1}\right) \cdot E_i = 1$, $(\tau p)^*\lambda \cdot E_i = \tau^*\lambda = g+1$, and $(\tau p)^*\delta_0 \cdot E_i = \tau^*\delta_0 = 6g+18$, for each $i = 1, \ldots, 2g-2$. Note how the intersection is independent of $\mu_{g,d,\boldsymbol{a}}$, as the pencil has zero intersection with the Brill-Noether class $\mathcal{BN}_g$ [Laz86]. It follows that the coefficients of the class $\left[\overline{\mathbb{H}}^{\boldsymbol{a}}_{g,d}\right]$ satisfy

$$2k(g+1)(g-1)^2\, \nu_{g,d,\boldsymbol{a}} = kc_\eta + (g+1)c_\lambda - (6g+18)c_0.$$

### 4.3.3 A pencil of hyperelliptic curves

We consider here a pencil of hyperelliptic curves, following [Fab90, pg. 361-363]. Let $S \to \mathbb{P}^2$ be a double cover branched over a general smooth curve of degree $2g+2$ in $\mathbb{P}^2$. This gives a two-dimensional family of hyperelliptic curves of genus $g$. Consider a general pencil of hyperelliptic curves in $S$. Such a pencil has two base points [Fab90, pg. 361-363]. Let $p\colon \mathcal{X} \to \mathbb{P}^1$ be the corresponding family of curves obtained by blowing up the two base points. A choice of $k(g-1)$ lines in $\mathbb{P}^2$ gives a $k$-canonical divisor on each curve in the family after pulling back to $\mathcal{X}$. One has

$$\deg \eta = k, \qquad \deg \lambda = \frac{g(g+1)}{2}, \qquad \deg \delta_0 = 2(g{+}1)(2g{+}1), \qquad \deg \delta_i = 0, \quad \text{for } i \geq 1.$$

The intersection with $\lambda$ and $\delta_i$ is computed as in [Fab90, pg. 361-363], and the intersection with $\eta$ can be computed as in §4.3.2.

Furthermore, let $C_i$, for $i = 1, \ldots, k(g-1)$, be the pullbacks to $\mathcal{X}$ of the $k(g-1)$ lines marking the $k$-canonical divisors, let $f$ be the class of a fiber of $p$, and $E_1, E_2$ the classes of the two exceptional divisors. Recall the pointed Brill-Noether class $\left[\mathcal{X}^{\boldsymbol{a}}_{g,d}\right]$ on $\mathcal{X}$ from (4.6). As in §4.3.2, the intersection with the divisor $\overline{\mathbb{H}}^{\boldsymbol{a}}_{g,d}$ equals

$$\begin{aligned}\left[\mathcal{X}^{\boldsymbol{a}}_{g,d}\right] \cdot \left(\sum_{i=1}^{k(g-1)} C_i\right) &= \left[\mathcal{X}^{\boldsymbol{a}}_{g,d}\right] \cdot k(g-1)(f + E_1 + E_2) \\ &= kg(g+1)(g-1)^2 \nu_{g,d,\boldsymbol{a}} - \frac{1}{3} k(g+1)(g-1)^2(g-2)\mu_{g,d,\boldsymbol{a}}.\end{aligned}$$

Here we used that the nonzero intersections are given by $c_1\left(\omega_{\mathcal{X}/\mathbb{P}^1}\right) \cdot f = 2g-2$, $c_1\left(\omega_{\mathcal{X}/\mathbb{P}^1}\right) \cdot E_i = 1$, $p^*\lambda \cdot E_i = \frac{g(g+1)}{2}$, and $p^*\delta_0 \cdot E_i = 2(g+1)(2g+1)$, for each $i = 1, 2$.

It follows that the coefficients of the class $\left[\overline{\mathbb{H}}_{g,d}^{\boldsymbol{a}}\right]$ satisfy

$$kg(g{+}1)(g{-}1)^2\nu_{g,d,\boldsymbol{a}} - \frac{1}{3}k(g{+}1)(g{-}1)^2(g{-}2)\mu_{g,d,\boldsymbol{a}} = kc_\eta + \frac{g(g+1)}{2}c_\lambda - 2(g{+}1)(2g{+}1)c_0.$$

#### 4.3.4 Branched covers in $g = 2$.

Consider fixed sections $\Gamma_1, \ldots, \Gamma_5 \sim (1,0)$ and $\Gamma_6 \sim (1,1)$ on $\mathbb{P}^1 \times \mathbb{P}^1$. Let $\mathrm{bl}\colon S \to \mathbb{P}^1 \times \mathbb{P}^1$ be the blow-up at the five points where $\Gamma_6$ meets one of the other $\Gamma_i$, and let $\pi\colon \mathcal{X} \to S$ be the (admissible) double cover of $S$ branched along the proper transform $\widetilde{\Gamma}_i$ of $\Gamma_i$, for $i = 1, \ldots, 6$. In this way, we obtain a family $p\colon \mathcal{X} \to \mathbb{P}^1$ whose fibers are genus 2 curves branched along the five fixed points indicated by $\Gamma_i$, for $1 \leq i \leq 5$, and also along the moving point indicated by $\Gamma_6$. Finally, we pick $k$-differentials on the fibers of $p$ by considering the pull-back of additional $k$ distinct, fixed sections $S_1, \ldots, S_k \sim (1,0)$ on $\mathbb{P}^1 \times \mathbb{P}^1$. We have

$$\deg \eta = \frac{k}{2}, \qquad \deg \lambda = 1, \qquad \deg \delta_0 = 10.$$

One can compute the degree of $\delta_0$ by noting that the family intersects $\delta_0$ five times above the intersection of $\Gamma_6$ with the other sections $\Gamma_i$, each time with multiplicity two (a result of the construction of these $g = 2$ curves as an admissible cover). We can find the degree of $\lambda$ by using the relation $\lambda = \frac{1}{10}\delta_0 + \frac{1}{5}\delta_1$ valid for $g = 2$ and the fact that $\deg \delta_1 = 0$. Finally, $\deg \eta$ is computed using the relation $\eta = \omega^{\otimes k} - \sum_{i=1}^{k}(S_{i,a} + S_{i,b})$, where $S_{i,a} + S_{i,b}$ is the preimage in $\mathcal{X}$ of the proper transform of $S_i$ in $S$. Intersecting the above relation with the section $Q$ corresponding to the ramification point above $\widetilde{\Gamma}_1$ yields

$$\deg \eta = Q \cdot \omega^{\otimes k} = -k\, Q^2 = -\frac{k}{2}\left(\widetilde{\Gamma}_1\right)^2 = \frac{k}{2}.$$

Since $\deg \overline{\mathbb{W}}_2^k = k$, with the intersections lying above the $k$ points where $\Gamma_6$ meets the sections $S_i$, this test curve gives the relation $k = \frac{k}{2}c_\eta + c_\lambda - 10c_0$ for $g = 2$.

## 4.4 The case $g = 2$

Here we extend the proof of Lemma 4.1 to the case $g = 2$.

*Proof of Lemma 4.1 when $g = 2$.* When $g = 2$ the only pointed Brill-Noether divisor is the Weierstrass divisor [EH89, Lemma 3.3]. Using the test curves described in §§4.3.1, 4.3.3, and 4.3.4, we compute $\overline{\mathbb{W}}_2^k = -6\eta + 9k\lambda - \frac{1}{2}k\delta_0$. Using $\lambda = \frac{1}{10}\delta_0 + \frac{1}{5}\delta_1$ when $g = 2$, one can see that this is equivalent to $\pi_*(\mathcal{Z}_1 \cdot \mathcal{W}_2)$. □

## 4.5 Strata for $g = 2$

The divisors $\overline{\mathbb{W}}_2^1$ and $\overline{\mathbb{W}}_2^2$ are precisely the divisorial strata $\overline{\mathbb{H}}_2^1(2)$ and $\overline{\mathbb{H}}_2^2(2,1,1)$, respectively. After [KSZ19, Def. 1.3, Thm 1.12], one has

$$\left[\overline{\mathbb{H}}_2^2(2,1,1)\right] + 2\left[\overline{\mathbb{H}}_2^2(2,2)\right] = 72\lambda - 10\eta - 6\delta_0 - 6\delta_1.$$

The computation of the class of $\overline{\mathbb{H}}_2^2(2,1,1)$ from Theorem 1.3 implies the following:

**Corollary 4.4.** *The stratum $\overline{\mathbb{H}}_2^2(2,2)$ parametrizing quadratic differentials which are squares of holomorphic differentials has class*

$$\left[\overline{\mathbb{H}}_2^2(2,2)\right] = -2\eta + 12\lambda - \delta_0 \in \operatorname{Pic}\left(\mathbb{P}\left(\mathbb{E}_2^2\right)\right).$$

For $k \geq 3$, the divisor $\overline{\mathbb{W}}_2^k$ no longer coincides with the divisorial stratum $\overline{\mathbb{H}}_2^k(2,1,\ldots,1)$, since the double zero of a $k$-differential need not be a Weierstrass point when $k \geq 3$.

# 5 Rigidity and extremality results

**Lemma 5.1** ([Che13, Lemma 4.1]). *Let $D$ be an irreducible effective divisor in a projective variety $X$, and let $S$ be a set of irreducible effective curves contained in $D$ such that $\bigcup_{C \in S} C$ is Zariski dense in $D$. If for every curve $C$ in $S$ one has*

$$C \cdot (D + B) \leq 0, \qquad \text{for a fixed big divisor class } B \text{ on } X,$$

*then $D$ is extremal in $\overline{\mathrm{Eff}}^1(X)$.*

Recall that a divisor class is big if it lies in the interior of the pseudo-effective cone. We emphasize that in Lemma 5.1, the curves in $S$ are not required to be moving in $D$. For convenience we will recall here the proof from [Che13].

*Proof of Lemma 5.1 from [Che13].* Suppose that $D$ is not extremal in $\overline{\mathrm{Eff}}^1(X)$, so that $D = D_1 + D_2$ where $D_1, D_2$ are pseudoeffective and not proportional to $D$. We may assume that $D_1$ and $D_2$ are on the boundary of $\overline{\mathrm{Eff}}^1(X)$ since otherwise we may replace $D_1$ and $D_2$ with the divisor classes in the intersection of the linear span $\langle D_1, D_2 \rangle$ and the boundary of $\overline{\mathrm{Eff}}^1(X)$. Note that this means $D_i - sD$ for $i = 1, 2$ is not pseudoeffective for any $s > 0$.

Since $B$ is big, $B = A + E$ with $A$ ample and $E$ effective with $E = bD + F$ for some $b \geq 0$. Thus,

$$\begin{aligned} C \cdot (D + B) &= C \cdot ((b+1)D + A + F) \\ &\implies C \cdot ((b+1)D + F) < 0 \end{aligned}$$

since $A$ is ample and so $C \cdot A > 0$. If $C \cdot F < 0$, then $C \subset F$. Such $C$ must form a non-dense subset of $S$, since $F$ has no $D$ components and S is dense in $D$.

Thus, we may assume that no $C$ in $S$ is contained in $F$. This assumption guarantees that $C \cdot D < 0$. With this assumption, since $C \cdot (D_1 + D_2) = C \cdot D < 0$, we may further assume, without loss of generality, that for some dense subset $S_1 \subseteq S$, we have $C \cdot D_1 \leq \frac{1}{2} \cdot (C \cdot D)$ for $C \in S_1$.

Let $F_n = nD_1 + B$ for $n$ sufficiently large. This divisor is in the interior of $\overline{\mathrm{Eff}}^1(X)$ since it is the sum of a pseudo-effective and a big divisor. Note that when $k < \frac{n}{2} - 1$, we have $C \cdot (F_n - kD) < 0$ for all $C \in S_1$. Moreover, since these curves $C \in S_1$ are dense in $D$, the multiplicity of $D$ in the base locus of $F_n$ is $\geq \frac{n}{2} - 1$ and the class $E_n = F_n - \left(\frac{n}{2} - 1\right) D$ is pseudo-effective. When $n \to \infty$, $\frac{1}{n} E_n \to D_1 - \frac{1}{2} D$, and so $D_1 - \frac{1}{2} D$ is pseudoeffective

□

We apply below Lemma 5.1 to deduce the extremality of the divisors $\overline{\mathbb{H}}_g^1(2, 1^{2g-4})$, $\overline{\mathbb{H}}_g^2(2, 1^{4g-6})$, and $\mathscr{Z}_1$ using the set $S$ of Teichmüller curves in these divisors. While such Teichmüller curves are not moving, their union is dense in these divisors.

*Proof of Theorem 1.5.* From [KZ11] we have that

$$\overline{\mathbb{H}}_g^1(2, 1^{2g-4}) = 24\lambda - (6g - 6)\eta - 2\delta_0 - 3 \sum_{i=1}^{\lfloor g/2 \rfloor} \delta_i.$$

Let $\mathcal{C}$ be the closure of a Teichmüller curve generated by some $(C, \omega) \in \overline{\mathbb{H}}_g^1(2, 1^{2g-4})$. Let $\chi = 2 - 2g(\mathcal{C}) - |\Delta| = -2\mathcal{C} \cdot \eta$ [Mö06], where $|\Delta|$ is the number of cusps in $\mathcal{C}$, and $L$ the sum of its first $g$ Lyapunov exponents. Here $\chi$ is the Euler characteristic of $\mathcal{C}$ (we note that a different sign convention is used for $\chi$ in [CM12]). We are concerned with the partition $\mu = (m_1, \ldots, m_n) = (2, 1^{2g-4})$. Using [CM12, Proposition 4.8] we have that

$$\mathcal{C}\cdot\lambda = -\frac{\chi}{2}L$$
$$\mathcal{C}\cdot\delta_0 = -\frac{\chi}{2}(12L - 12\kappa_\mu)$$

where $\kappa_\mu = \frac{1}{12}\sum_{j=1}^{n}\frac{m_j(m_j+2)}{m_j+1}$. Since Teichmüller curves do not intersect higher boundary divisors (see [CM12, Corollary 3.2]),

$$\mathcal{C}\cdot\delta_i = 0 \;\text{ for }\; i > 0.$$

So,

$$\mathcal{C}\cdot\overline{\mathbb{H}}_g^1(2,1^{2g-4}) = \frac{\chi}{3}.$$

Moreover, by [CM12, Proposition 4.8]

$$\mathcal{C}\cdot\psi_i = \frac{\mathcal{C}\cdot\lambda - (\mathcal{C}\cdot\delta)/12}{(m_i+1)\kappa_\mu} = -\frac{\chi}{2(m_i+1)}.$$

Since $\psi_i$ has positive degree on nonconstant families [HM98, Chapter 6] $\chi < 0$. Now let $A$ be an ample divisor in $\mathbb{P}\left(\mathbb{E}_g^1\right)$. We write

$$A = a\lambda + b\eta + \sum_{i=0}^{\lfloor g/2\rfloor} c_i\delta_i.$$

We must now choose a sufficiently small value $d$ such that $\mathcal{C}\cdot(\overline{\mathbb{H}}_g^1(2,1^{2g-4}) + dA) \leq 0$ for all Teichmüller curves in $\overline{\mathbb{H}}_g^1(2,1^{2g-4})$. Let

$$d = \inf_{\substack{\text{Teichmüller curves}\\ \text{in } \overline{\mathbb{H}}_g^1(2,1^{2g-4})}}\left\{\frac{2}{3(b - 12c_0\kappa_\mu + (a+12c_0)L)}\right\}.$$

The expression in the brackets comes from solving for $d$ in $\mathcal{C} \cdot (\overline{\mathbb{H}}_g^1(2, 1^{2g-4}) + dA) = 0$ using the intersection information given above. Since $\mathcal{C} \cdot A > 0$ ($A$ is ample) and $\mathcal{C} \cdot \overline{\mathbb{H}}_g^1(2, 1^{2g-4}) < 0$ for all Teichmüller curves $\mathcal{C}$, the expression in the brackets will always be positive and will only depend on $L$. Moreover, the infimum may never be zero since the sum of Lyapunov exponents $L$ has a uniform upper bound $g$. Since Teichmüller curves in any stratum are Zariski dense, we have shown that $\overline{\mathbb{H}}_g^1(2, 1^{2g-4})$ is extremal by Lemma 5.1.

The divisor class of $\overline{\mathbb{H}}_g^2(2, 1^{4g-6})$, computed in [KZ13], is

$$\overline{\mathbb{H}}_g^2(2, 1^{4g-6}) = 72\lambda - 10(g-1)\eta - 6\sum_{i=0}^{\lfloor g/2 \rfloor} \delta_i$$

for $g \geq 3$. Here we denote the partition $(d_1, \ldots, d_n) = (2, 1^{4g-6})$. Let $\mathcal{C}$ be a Teichmüller curve generated by a half translation surface $(C, q) \in \overline{\mathbb{H}}_g^2(2, 1^{4g-6})$. Let $L^+$ be the sum of the involution invariant Lyapunov exponents (see [EKZ14] and [CM14, Section 2.2] for background material) and let $\chi = 2 - 2g(\mathcal{C}) - |\Delta| = -\mathcal{C} \cdot \eta$ [MÖ6]. From [EKZ14] we can write

$$L^+ = c_{\text{area}} + \kappa_d, \quad \text{where} \quad \kappa_d = \frac{1}{24}\sum_{j=1}^{n} \frac{d_j(d_j+4)}{d_j+2}$$

and $c_{\text{area}}$ is the area Siegel-Veech constant of $(C, q)$. By [CM14, Proposition 4.2]

$$\mathcal{C} \cdot \lambda = -\frac{\chi}{2}(c_{\text{area}} + \kappa_d)$$
$$\mathcal{C} \cdot \delta = -6\chi \cdot c_{\text{area}}.$$

Hence,

$$\mathcal{C} \cdot \overline{\mathbb{H}}_g^2(2, 1^{4g-6}) = \frac{\chi}{2}.$$

When $g = k = 2$, we have $\overline{\mathbb{H}}_2^2(2,1^2) = 18\lambda - 6\eta - \delta_0 = 28\lambda - 6\eta - 2\delta_0 - 2\delta_1$ where the second equality comes from the $g = 2$ relation $\lambda = \frac{1}{10}\delta_0 + \frac{1}{5}\delta_1$. Performing the same intersections as above gives us that $\mathcal{C} \cdot \overline{\mathbb{H}}_2^2(2,1^2) = -\chi(2c_{\text{area}} - \frac{83}{36})$. By [EKZ14, Corollary 3] we know that the sum of the involution anti-invariant Lyapunov exponents $L^- = 4/3$. By Theorem 2 of the same paper, we know that $L^- - L^+ = 1/6$ which implies $L^+ = 7/6$. Since $L^+ = c_{\text{area}} + \kappa_d$, and $\kappa_d = 19/72$, we get that $c_{\text{area}} = 65/72$. Thus, $\mathcal{C} \cdot \overline{\mathbb{H}}_2^2(2,1^2) = -\chi(2c_{\text{area}} - \frac{83}{36}) = \chi/2$ as well.

If $a\lambda + b\eta + \sum_{i=0}^{\lfloor g/2 \rfloor} c_i\delta_i$ is an ample divisor, we can ensure that all coefficients for the boundary divisors are the same by adding on an appropriate effective divisor of boundary divisors. This gives us a big divisor

$$A = a\lambda + b\eta + c\delta$$

where $c = \max_i\{c_i\}$. Note that when $\mathcal{C} \cdot A \leq 0$ and $B = dA$ any $d > 0$ satisfies the condition in Lemma 5.1. So assuming $\mathcal{C} \cdot A > 0$, we set

$$d = \inf_{\substack{\text{Teichmüller curves} \\ \text{in } \overline{\mathbb{H}}_g^2(2,1^{4g-6}) \text{ with } \mathcal{C}\cdot A>0}} \left\{ \frac{1}{2b + 12c_{\text{area}}c + aL^+} \right\}.$$

The expression in the brackets comes from solving for $d$ in the expression

$$\mathcal{C} \cdot (\overline{\mathbb{H}}_g^2(2,1^{4g-6}) + dA) = 0.$$

Since $c_{\text{area}}$ is bounded from above and $L^+ = c_{\text{area}} + \kappa_d$, $d$ is positive and so $\overline{\mathbb{H}}_g^2(2,1^{4g-6})$ is extremal by Lemma 5.1.

In what follows we will use the shorthand $\overline{\mathbb{H}}_g^k(2)$ for the closure of the locus of $k$-differentials with a double zero. The argument for the rigidity of $\overline{\mathbb{H}}_g^k(2)$ is the same for the cases $k \in \{1,2\}$. Suppose that there exists an effective divisor $D$ such that

$D \equiv m\overline{\mathbb{H}}_g^k(2)$, for some $m > 0$. We may assume that $D$ does not contain $\overline{\mathbb{H}}_g^k(2)$, since otherwise we could simply consider the divisor $D \setminus \overline{\mathbb{H}}_g^k(2)$ and reduce the coefficient $m$. For all Teichmüller curves $\mathcal{C}$ in $\overline{\mathbb{H}}_g^k(2)$, we have that $\mathcal{C} \cdot \overline{\mathbb{H}}_g^k(2) < 0$, and thus $\mathcal{C} \cdot D < 0$. This means that $D$ contains the entire collection $S$ of such Teichmüller curves. Since $S$ is dense in $\overline{\mathbb{H}}_g^k(2)$, we deduce that $D$ contains $\overline{\mathbb{H}}_g^k(2)$, which contradicts the assumption. □

For completeness we include the following proposition.

**Proposition 1.** *The boundary divisors $\delta_i$, $0 \leq i \leq \lfloor g/2 \rfloor$, span extremal rays in $\overline{\mathrm{Eff}}^1(\mathbb{P}(\mathbb{E}_g^k))$.*

We will use the following well-known condition to check the extremality of the boundary divisors. An irreducible curve $C$ in a projective variety $X$ is called a *moving curve* if it is a member of an algebraic family covering an open dense subset of $X$.

**Lemma 5.2** ([CC14, Lemma 4.1])**.** *Suppose that $C$ is a moving curve in an irreducible effective divisor $D$ of a projective variety $X$. Suppose that $C$ satisfies $C \cdot D < 0$. Then $D$ is extremal.*

*Proof of Proposition 4.2.* Let $f : \mathbb{P}(\mathbb{E}_g^k) \to \overline{\mathcal{M}}_g$. We will use the following strategy. We will find moving curves in each of the irreducible boundary divisors of $\overline{\mathcal{M}}_g$ which satisfy the condition of Lemma 5.2. Then let $C$ be such a moving curve and let $D$ be an irreducible boundary divisor in $\overline{\mathcal{M}}_g$. Given a point in $f^{-1}(C)$, we can find a curve $C'$ through it by taking the intersection of $g-1$ hyperplane classes $H_1, \ldots, H_{g-1}$ in $f^{-1}(C)$. When the choice of these hyperplane classes is general, $C'$ is irreducible by Bertini's theorem and moreover $C'$ covers $C$. To see the latter statement, note that as a result of the irreducibility of $C'$ we just need to show that the image of $C'$ under

$f$ is not a single point $p \in C$. Note that $f^{-1}(p)$ is a divisor in $f^{-1}(C)$ and so it must intersect $\cap_{i=1}^{g-1} H_i$ positively. Thus $C'$ covers $C$. Finally, since varying the hyperplane classes used to construct $C'$ will not change the numerical equivalence class of $C'$, we know that it is indeed a moving curve. Thus, $f^*D \cdot C' = D \cdot f_*C' < 0$ and we can conclude by Lemma 5.2 that $f^*D$ is extremal. All that remains is to find appropriate moving curves in each of the boundary divisors of $\overline{\mathcal{M}}_g$.

Let $X$ be the following curve in $\delta_0 \subset \overline{\mathcal{M}}_g$: take a genus $g-1$ curve $C$ and identify a fixed point $p$ of $C$ to a varying point $q$ of $C$. This is a moving curve in $\delta_0$ and

$$X \cdot \delta = \deg(N_{\widetilde{\Delta}/S} \otimes N_{\widetilde{C \times p}/S}) = \Delta^2 - 1 = 3 - 2g$$

where $S$ is the blow up of $C \times C$ at $(p, p)$ and $\widetilde{\Delta}$ and $\widetilde{C \times p}$ denote the proper transforms. Since $X \cdot \delta_1 = 1$, we also have $X \cdot \delta_0 = 2 - 2g < 0$.

Now assume that $g \geq 3$ and let $X$ be the moving curve in $\delta_i \subset \overline{\mathcal{M}}_g$ given by attaching a general genus $i$ curve $C_1$ to a general genus $g - i$ curve $C_2$ and varying the point of attachment in $C_2$. In the computation for test curve $C$ in the proof of Theorem 1.1 we explained that $X \cdot \delta_i = 2 - 2(g - i) < 0$. When $g = 2$, we can choose our moving curve $X$ in $\delta_1$ to be the family given by attaching a pencil of plane cubics to a general genus 1 curve. In the computation for test curve $B$ in the proof of Theorem 1.1, we explained that $X \cdot \delta_1 = -1$. Thus, we have found all necessary moving curves. □

To prove Theorem 1.6 we again make use of a dense family of Teichmüller curves.

*Proof of Theorem 1.6.* We first deal with the $k = 1$ case. Let $\mathcal{C}$ be the closure of a Teichmüller curve generated by some element $(C, \mu, P)$ in $\mathcal{Z}_1 \cap \mathbb{H}^1_{g,1}(1^{2g-2})$. This is the lift of the closure of a Teichmüller curve in $\overline{\mathbb{H}}^1_g(1^{2g-2})$ obtained by marking a point in the support of the canonical divisor. The collection $S$ of such Teichmüller curves $\mathcal{C}$ is dense in $\mathcal{Z}_1 \cap \overline{\mathbb{H}}^1_{g,1}(1^{2g-2})$, hence it is dense in $\mathcal{Z}_1$. As before, from [CM12, §4], one has

$$\mathcal{C} \cdot \lambda = -\frac{\chi}{2}L, \qquad \mathcal{C} \cdot \delta_0 = -\frac{\chi}{2}(12L - 3g + 3), \qquad \mathcal{C} \cdot \omega = 2\,\frac{\mathcal{C} \cdot \lambda - \mathcal{C} \cdot \delta_0/12}{g-1} = -\frac{\chi}{4}, \tag{5.1}$$

and $\mathcal{C} \cdot \delta_i = 0$, for $i \geq 1$. From [CM12], one has $\mathcal{C} \cdot \eta = -\frac{\chi}{2}$. Combining this with Lemma 3.1, we have that

$$\mathcal{C} \cdot \mathcal{Z}_1 = \mathcal{C} \cdot (\omega - \eta) = \frac{\chi}{4}.$$

Since $\omega$ is ample on any nonconstant family not all of whose elements are singular [HM98, Thm 6.33], one has $\mathcal{C} \cdot \omega > 0$. We deduce $\chi < 0$ and thus $\mathcal{C} \cdot \mathcal{Z}_1 < 0$. In the following we show that we can apply Lemma 5.1 with a certain big divisor class $B$. Let

$$A := c_\eta\, \eta + c_\lambda\, \lambda + c_\omega\, \omega + \sum_{i=0}^{g-1} c_i\, \delta_i$$

be an ample divisor class on $\mathbb{P}\left(\mathbb{E}^1_{g,1}\right)$, and define $B := dA$ for a sufficiently small value $d$ so that $\mathcal{C} \cdot (\mathcal{Z}_1 + B) \leq 0$ for all $\mathcal{C} \in S$. Namely, let

$$d := \inf_{\mathcal{C} \in S} \left\{ \frac{1}{2c_\eta + c_\omega - 6(g-1)c_0 + 2L(c_\lambda + 12c_0)} \right\}.$$

The expression in the brackets comes from solving for $d$ in $\mathcal{C} \cdot (\mathcal{Z}_1 + dA) = 0$ for a given Teichmüller curve, and it is positive since $\mathcal{C} \cdot \mathcal{Z}_1 < 0$ and $\mathcal{C} \cdot A > 0$. Furthermore, the value in the brackets depends only on the sum of Lyapunov exponents $L$. Since $L \leq g$ (see [Zor06]), the infimum here will indeed be positive, and thus the class $B = dA$ is big. The extremality of $\mathcal{Z}_1$ follows by applying Lemma 5.1 with such a $B$.

We now consider the case $k = 2$. The proof proceeds as in the previous case. Let $\mathcal{C}$ be the closure of a Teichmüller curve generated by some marked half-translation surface $(C, \mu, P)$ in $\mathcal{Z}_1 \cap \mathbb{H}^2_{g,1}(1^{4g-4})$ obtained by marking a point in the support of the quadratic differential as in [CM14, §4.1]. The collection $S$ of such Teichmüller curves $\mathcal{C}$ is dense in $\mathcal{Z}_1 \cap \overline{\mathbb{H}}^2_{g,1}(1^{4g-4})$ [Che11], hence it is dense in $\mathcal{Z}_1$. Again from [CM14, Prop. 4.2], one has

$$\mathcal{C} \cdot \lambda = -\frac{\chi}{36}\big(18c_{\text{area}} + 5(g-1)\big), \qquad \mathcal{C} \cdot \delta = -6\chi\, c_{\text{area}}, \qquad \mathcal{C} \cdot \omega = -\frac{\chi}{3}, \tag{5.2}$$

and from [MÖ6] one also has $\mathcal{C} \cdot \eta = -\chi$. From Lemma 3.1, we have that

$$\mathcal{C} \cdot \mathcal{Z}_1 = \mathcal{C} \cdot (2\omega - \eta) = \frac{\chi}{3}.$$

Using the same argument as before shows that $\mathcal{C} \cdot \mathcal{Z}_1 < 0$. Now take some ample divisor class

$$c_\eta \eta + c_\lambda \lambda + c_\omega \omega + \sum_{i=0}^{g-1} c_i \delta_i \quad \text{on } \overline{\mathbb{H}}^2_{g,1}(1^{4g-4}),$$

and let $c_\delta := \max_i c_i$. Consider the big divisor class $A = c_\eta \eta + c_\lambda \lambda + c_\omega \omega + c_\delta \delta$. We define $B := dA$ for some sufficiently small value $d$ so that $\mathcal{C} \cdot (\mathcal{Z}_1 + B) \leq 0$ for all $\mathcal{C} \in S$. For this, let

$$d := \inf_{\mathcal{C} \in S} \left\{ \frac{12}{36c_\eta + 12c_\omega + 5(g-1)c_\lambda + c_{\text{area}}(18c_\lambda + 216c_\delta)} \right\}.$$

Since the constant $c_{\text{area}}$ is bounded from above, $d$ is positive, and thus $B$ is big. From Lemma 5.1, $\mathcal{Z}_1$ is extremal when $k = 2$ as well.

The argument for the rigidity of $\mathcal{Z}_1$ is the same for the cases $k \in \{1, 2\}$. Suppose that there exists an effective divisor $D$ such that $D \equiv m\mathcal{Z}_1$, for some $m > 0$. We may assume that $D$ does not contain $\mathcal{Z}_1$, since otherwise we could simply consider the divisor $D \setminus \mathcal{Z}_1$ and reduce the coefficient $m$. For all Teichmüller curves $\mathcal{C}$ in $\overline{\mathbb{H}}^k_{g,1}\left(1^{k(2g-2)}\right)$, we have that $\mathcal{C} \cdot \mathcal{Z}_1 < 0$, and thus $\mathcal{C} \cdot D < 0$. This means that $D$ contains the entire collection $S$ of such Teichmüller curves. Since $S$ is dense in $\mathcal{Z}_1$, we deduce that $D$ contains $\mathcal{Z}_1$, which contradicts the assumption. □

## 5.1 Rigidity and extremality for $n > 2$

After Theorem 1.6, $[\mathcal{Z}_1]$ is rigid and extremal in $\overline{\mathrm{Eff}}^1\left(\mathbb{P}\left(\mathbb{E}^k_{g,1}\right)\right)$ for $k \in \{1, 2\}$. For the following statement, it is enough to assume the *weaker* condition that $[\mathcal{Z}_{n-1}]$ is rigid and extremal in $\mathrm{Eff}^{n-1}\left(\mathbb{P}\left(\mathbb{E}^k_{g,n-1}\right)\right)$. Moreover, the statement is valid for arbitrary $k$:

**Proposition 5.3.** *For $n > 2$, if $[\mathcal{Z}_{n-1}]$ is rigid and extremal in $\mathrm{Eff}^{n-1}\left(\mathbb{P}\left(\mathbb{E}^k_{g,n-1}\right)\right)$, then $[\mathcal{Z}_n]$ is rigid and extremal in $\mathrm{Eff}^n\left(\mathbb{P}\left(\mathbb{E}^k_{g,n}\right)\right)$, provided that $\mathcal{Z}_n$ is non-empty, i.e., $n \leq k(2g-2)$.*

*Proof.* For the extremality, assume the class of $\mathcal{Z}_n$ can be expressed as

$$[\mathcal{Z}_n] = \sum_{\alpha} c_\alpha [E_\alpha] \quad \in \mathrm{Eff}^n\left(\mathbb{P}\left(\mathbb{E}^k_{g,n}\right)\right), \tag{5.3}$$

where $c_\alpha > 0$ and $E_\alpha$ is an irreducible locus of codimension $n$ in $\mathbb{P}\left(\mathbb{E}^k_{g,n}\right)$, for each $\alpha$. After rearranging and rescaling, we can assume that none of the classes $[E_\alpha]$ is proportional to $[\mathcal{Z}_n]$. Since $\pi_*[\mathcal{Z}_n] \neq 0$, where $\pi\colon \mathbb{P}\left(\mathbb{E}^k_{g,n}\right) \to \mathbb{P}\left(\mathbb{E}^k_{g}\right)$ is the forgetful map, there exists at least one class on the right-hand side, say $[E_0]$, such that $\pi_*[E_0] \neq 0$. In particular, one has $(\pi_i)_*[E_0] \neq 0$, where $\pi_i\colon \mathbb{P}\left(\mathbb{E}^k_{g,n}\right) \to \mathbb{P}\left(\mathbb{E}^k_{g}\right)$ forgets the $i$-th marked point, for each $i = 1, \ldots, n$.

Applying $(\pi_i)_*$ to both sides of (5.3), one has

$$(k(2g-2)-(n-1))\,[\mathcal{Z}_{n-1}] = \sum_{\alpha} c_\alpha\,(\pi_i)_*\,[E_\alpha] \,\in\, \mathrm{Eff}^{n-1}\left(\mathbb{P}\left(\mathbb{E}^k_{g,n-1}\right)\right).$$

Since $[\mathcal{Z}_{n-1}]$ is assumed to be extremal in $\mathrm{Eff}^{n-1}\left(\mathbb{P}\left(\mathbb{E}^k_{g,n-1}\right)\right)$, the non-zero class $(\pi_i)_*\,[E_0]$ is necessarily proportional to $[\mathcal{Z}_{n-1}]$. Since $[\mathcal{Z}_{n-1}]$ is also assumed to be rigid, it follows that $(\pi_i)_*\,[E_0]$ is supported on $\mathcal{Z}_{n-1}$, hence $E_0 \subset \pi_i^{-1}\mathcal{Z}_{n-1}$, for each $i$. We conclude that $E_0 \subset \bigcap_{i=1}^n \pi_i^{-1}\mathcal{Z}_{n-1}$. This implies that for a general element $(C, \mu, P_1, \ldots, P_n)$ in $E_0$, each subset of size $n-1$ of the set of marked points $P_1, \ldots, P_n$ consists of *distinct* points in the support of $\mu$. Since $n > 2$, this implies that the $n$ marked points are assigned to distinct points in the support of $\mu$, hence $E_0$ coincides with $\mathcal{Z}_n$, a contradiction.

For the rigidity, suppose that there is some other effective class $E$ such that $E \sim m\mathcal{Z}_n$. We can assume that $E$ is not supported on $\mathcal{Z}_n$, otherwise we could reduce the coefficient $m$. Since $(\pi_i)_*(E) \sim (k(2g-2)-(n-1))\,\mathcal{Z}_{n-1}$ for $i \in \{1, \ldots, n\}$ and $\mathcal{Z}_{n-1}$ is rigid, we know that $(\pi_i)_*(E)$ must be supported on $\mathcal{Z}_{n-1}$. This means that $E \subset \bigcap_{i=1}^n \pi_i^{-1}\mathcal{Z}_{n-1}$ and thus must be $\mathcal{Z}_n$. Thus we arrive at a contradiction. □

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