Upsilon Invariant, Fibered Knots and Right-veering Open Books

Author: Dongtai He

Persistent link: http://hdl.handle.net/2345/bc-ir:108214

This work is posted on eScholarship@BC, Boston College University Libraries.

Boston College Electronic Thesis or Dissertation, 2018

Copyright is held by the author, with all rights reserved, unless otherwise noted.
Upsilon Invariant, Fibered Knots and Right-veering Open Books

Dongtai He

A dissertation

submitted to the Faculty of

the Department of Mathematics

in partial fulfillment

of the requirements for the degree of

Doctor of Philosophy

Boston College
Morrissey College of Arts and Sciences
Graduate School

December 2018
Ozsváth, Stipsicz and Szabó define a one-parameter family \( \{ \Upsilon_K(t) \}_{t \in [0,2]} \) of Heegaard Floer knot invariants for knots \( K \subset S^3 \). We generalize \( \Upsilon_K(t) \) to knots in any rational homology sphere. We study the \( \Upsilon \)-invariant of a fibered knot. We prove that the \( \Upsilon \)-invariant can never reach its minimum slope if the monodromy of the fibration is not right-veering.
# Contents

1 Introduction ................................................................. 1  
1.1 Structure of this thesis ................................................. 4

2 Generalized $\Upsilon$– Invariant ................................. 5  
2.1 Knot Floer complex ......................................................... 5
2.2 The Definition and Properties of the $\Upsilon$–invariant .......... 8  
   2.2.1 t-filtration and $\Upsilon$ ............................................ 8
   2.2.2 $\Upsilon$ as a function of t ......................................... 9

3 The $\Upsilon$–invariant of Fibered Knots ......................... 12  
3.1 Right-veering diffeomorphism ......................................... 12
3.2 Knot Floer homology of fibered knots ............................ 13
3.3 Proof of Theorem 1.0.4 ................................................... 15
3.4 Examples ................................................................. 18

Bibliography ................................................................. 19
Acknowledgments

This is part of the author’s doctorate assignment. The author wishes to thank his advisor J. Elisenda Grigsby for invaluable mentoring. We would like to thank John Baldwin, Peter Feller, Joshua Greene for their comments and feedback.
To my parents
Chapter 1

Introduction

In [OSS17], Ozsváth, Stipsicz and Szabó define a one-parameter family \( \{ \Upsilon_K(t) \}_{t \in [0,2]} \) of Heegaard Floer knot invariants. \( \Upsilon_K(t) \) is a knot concordance invariant. It bounds the 4-ball genus:

\[
|\Upsilon_K(t)| \leq \gamma_4(K)t.
\]

Furthermore, they apply \( \Upsilon_K(t) \) to the smooth concordance group \( \mathcal{C} \). As an example, they show that the torus knot \( T_{3,4} \) is linearly independent to any alternating knot in \( \mathcal{C} \). In [OSS15], the authors prove that \( \Upsilon_K(1) \) gives a lower bound for the smooth 4-dimensional crosscap number of \( K \).

In this thesis, we generalize the \( \Upsilon \)–invariant to knots in rational homology spheres. For each \( \text{Spin}^c \)–structure \( s \), we define the invariant \( \Upsilon_{K,s}(t) \). Then we focus on the special case when \( K \) is a fibered knot.

In a similar setting, Grigsby, Licata and Wehrli [GLW16] define a family of annular Rasmussen invariants \( \{ d_t(L,o) \}_{t \in [0,2]} \) from the Khovanov-Lee complex of an oriented link in a thickened annulus. In particular, the authors study the case when \( (L,o) \) is a
braid closure $\hat{\beta}$ equipped with its braid-like orientation. They find a rather interesting connection between $d_t(\hat{\beta})$ and the positivity of braids:

**Theorem 1.0.1.** [GLW16] Let $\hat{\beta}$ be a braid closure with its natural orientation. If $\beta$ is quasipositive, then $d_t'(\hat{\beta}) = b$ for all $t \in [0,1)$, where $b$ is the braid index of $\beta$.

**Theorem 1.0.2.** [GLW16] If $d_t'(\hat{\beta}) = b$ for some $t \in [0,1)$, then $\beta$ is right-veering.

Inspired by the above theorems, the slope of the $\Upsilon$–invariant for fibered knots is of particular interest. Let $Y$ be a rational homology sphere and let $K \subset Y$ be a fibered knot. The fibered surface $\Sigma$ and the monodromy $\phi: \Sigma \to \Sigma$ define an open book decomposition $(\Sigma, \phi)$ on $Y$. By Giroux correspondence [Gir02], there is a one-to-one correspondence between open book decomposition up to positive stabilization and isotopy classes of contact structures $\xi$ on $Y$. $\xi$ induces a $\text{Spin}^c$ structure $s = s(\xi)$ on $Y$.

Since the work of Honda, Katez and Matić [HKM07], the notion of right-veering (definition 3.1.1) of the monodromy $\phi$ plays a vital role in contact geometry due to the following theorem [HKM07]:

**Theorem 1.0.3.** If $\xi$ is tight, then every open book $(\Sigma, \phi)$ compatible with $\xi$ is right-veering.

Ozsváth and Szabó define the contact invariant in Heegaard Floer homology in [OS05]. The invariant is a class $c(\xi) \in \widehat{HF}(-Y, s(\xi))$ assigned to a contact structure $\xi$ on $Y$. If $c(\xi) \neq 0$, then $\xi$ is tight. It follows from theorem 1.0.3 that any open book compatible with $\xi$ is right-veering. $c(\xi)$ does not detect right-veeringness completely, however; Honda, Katez and Matić prove that any contact structure admits a right-veering open book via positive stabilization [HKM07]. Moreover, Lisca [Lis11] shows
that it is possible to have an overtwisted contact structure compatible with a right-
veering open book which can not be destabilized. The following theorem attempts to
further extract information from the knot Floer complex of the binding $K$ by studying
$\Upsilon_{K,s(\xi)}(t)$.

**Theorem 1.0.4.** If $\Upsilon'_{K,s}(t) = -g$ for some $t \in [0,1)$, where $g$ is the genus of the
fibered surface $\Sigma$, then $\phi : \Sigma \to \Sigma$ is right-veering. The converse does not hold in
general.

This theorem is similar to theorem 1.0.2. However, the analogue of theorem 1.0.1
does not hold, as the $\Upsilon-$invariant doesn’t necessarily have a single slope on $t \in [0,1)$
when $\phi$ is a product of positive Dehn twist. Indeed, let $K$ be the torus knot $T(3,7)$,
then $\Upsilon_K(t) = -6t$ for $t \in [0, \frac{2}{3}]$ and $-4$ for $t \in [\frac{2}{3}, 1)$.

**Remark.** A result of Hedden [Hed05] tells us that given a fibered knot $K \subset S^3$, the
following are equivalent:

1. $K$ is strongly quasi-positive;

2. $\tau(K) = g(K)$;

3. the fibration is compatible with the unique tight contact structure on $S^3$.

1 and 2 combined with the fact that $\Upsilon_K(t) = -\tau(K)t$ [OSS17] at $t = 0$ show that
$\Upsilon_K(t) = -gt$ at $t = 0$, so the monodromy is right-veering, which also follows from
3. Unfortunately, we are unable to find any example such that $\Upsilon'_K(t) \neq -g(K)$ at 0
and $\Upsilon'_K(t) = -g(K)$ for some $t \in (0,1)$. Such an example will provide a fibered knot
with right-veering monodromy but supports overtwisted contact structure.
1.1 Structure of this thesis

The remainder of this thesis is organized as follows. In chapter 2 we first briefly review the construction of the Knot Floer complex. We focus on definitions and constructions that are necessary for our purpose. Then we define the generalized $\Upsilon$—invariant and establish some basic properties of $\Upsilon$. In chapter 3 we review the definition of right-veeringness and study the case for fibered knots. Then we prove theorem 1.0.4 and provide some examples.
Chapter 2

Generalized $\Upsilon$– Invariant

2.1 Knot Floer complex

In this section we briefly review the construction of the Heegaard Floer complex of knots following [OS04] and [Ras03]. Let $Y$ be a rational homology sphere, and let $K \subset Y$ a null-homologous knot. We can associate to the pair $(Y, K)$ a 2-pointed Heegaard diagram $(\Sigma, \alpha, \beta, w, z)$ consisting of the following data:

- A Heegaard surface of genus $g$, splitting $Y$ into two handlebodies $U_0$ and $U_1$;
- linearly independent curves $\alpha = \{\alpha_1, ..., \alpha_g\}$, $\beta = \{\beta_1, ..., \beta_g\}$ on $\Sigma$;
- Based points $w, z \in \Sigma - \alpha_1 - ... - \alpha_g - \beta_1 - ... - \beta_g$.

Connect $w$ and $z$ by a curve $a$ in $\Sigma - \alpha_1 - ... - \alpha_g$ and another curve in $\Sigma - \beta_1 - ... - \beta_g$. The knot $K$ is obtained by pushing $a$ and $b$ into $U_0$ and $U_1$ respectively. One can always construct such a 2-pointed diagram from a suitable Morse function on the knot complement.
Let $\Sigma^g$ be the Cartesian product of $g$ copies of $\Sigma$. The symmetric product $Sym^g(\Sigma)$ is obtained from $\Sigma^g$ quotient by the symmetric group $S_g$, which acts on $\Sigma^g$ by permutation. In other words $Sym^g(\Sigma)$ consists of unordered $g$-tuples of points in $\Sigma$. Inside $Sym^g(\Sigma)$ there are two half-dimensional tori:

$$T_\alpha = \alpha_1 \times \ldots \times \alpha_g / S_g, \quad T_\beta = \beta_1 \times \ldots \times \beta_g / S_g$$

A complex structure on $\Sigma$ induces one on $Sym^g(\Sigma)$, where $T_\alpha$ and $T_\beta$ are totally real. Let $x, y \in T_\alpha \cap T_\beta$ be two intersection points, and let $\pi_2(x, y)$ be the set of relative homotopy classes of disks

$$u : D^2 \to Sym^g(\Sigma),$$

with $u(-1) = x$, $u(1) = y$, and the lower half of $\partial D^2$ mapping to $T_\alpha$ and the upper half to $T_\beta$. For each $\phi \in \pi_2(x, y)$, let $\mathcal{M}(\phi)$ be the moduli space of $J$-holomorphic representatives of $\phi$, where $J$ is an almost complex structure on $Sym^g(\Sigma)$. $\mathcal{M}(\phi)$ admits an $\mathbb{R}$–action, and we denote the quotient space by $\hat{\mathcal{M}}(\phi)$. The dimension of $\hat{\mathcal{M}}(\phi)$ is called the Maslov index $\mu(\phi)$.

Let $C(K)$ be the free abelian group generated by intersection points $x \in T_\alpha \cap T_\beta$. $C(K)$ has two gradings: the Maslov (homological) grading and Alexander grading. Let $n_w(\phi) = \#\phi^{-1}(\{w\} \times Sym^{g-1}(\Sigma))$ and $n_z(\phi) = \#\phi^{-1}(\{z\} \times Sym^{g-1}(\Sigma))$. $n_w(\phi), n_z(\phi)$ are well-defined since $\{w\} \times Sym^{g-1}(\Sigma)$ and $\{z\} \times Sym^{g-1}(\Sigma)$ are both disjoint from $T_\alpha$ and $T_\beta$. The Alexander grading $A(x)$ is characterized by:

- the function $A(x) - A(y) = n_z(\phi) - n_w(\phi)$;
• the Euler characteristic $\Delta_K(T) = \sum a \sum_m (-1)^m \text{rank}(H_{a,m}(K))T^m = \Delta_K(T^{-1})$,

where $a$ is the Alexander grading and $m$ is the Maslov grading.

Now we can define the knot Floer complex $\mathcal{CFK}^\infty(Y,K)$:

• over $\mathbb{F}_2[U, U^{-1}]$,

• whose generators are elements of the form $[x, i, j]$, where $j - i$ is the Alexander grading of $x$,

• whose differential is given by

$$\partial^\infty [x, i, j] = \sum_{y \in \mathcal{T}_a \cap \mathcal{T}_\beta} \sum_{\phi \in \pi_2(x, y) | \mu(\phi) = 1} \#(\hat{M}(\phi))[y, i - n_w(\phi), j - n_z(\phi)]$$

where $\#(\hat{M}(\phi))$ is counted modulo 2,

• with $U$-action $U([x, i, j]) = [x, i - 1, j - 1]$,

• splitting as a direct sum:

$$\mathcal{CFK}^\infty(Y,K) = \bigoplus_{s \in \text{Spin}^c(Y)} \mathcal{CFK}^\infty(Y,K,s)$$

where $s$ runs over $\text{Spin}^c$ structures on $Y$.

The homology of $\mathcal{CFK}^\infty(Y,K,s)$ is $H\mathcal{F}^\infty(Y,s) \cong \mathbb{F}[U, U^{-1}]$ as a relatively graded $\mathbb{F}[U, U^{-1}]$-module. An absolute grading can be defined where the base element $1 \in \mathbb{F}[U, U^{-1}]$ has Maslov (homological) grading $d(Y,s)$, which is the Heegaard Floer correction term [OS03]. The $U$-action changes the Maslov grading by $-2$. There is a $\mathbb{Z} \oplus \mathbb{Z}$ filtration on $\mathcal{CFK}^\infty(Y,K,s) = C$ given by the map $[x, i, j] \mapsto [i, j]$, where $(i, j)$ corresponds to the algebraic and Alexander filtration respectively. $i = 0$ is the minimum algebraic filtration level such that the image of the inclusion induced map
on homology \( H(C\{i \leq k\}) \hookrightarrow H(C) \) contains the base element of degree \( d(Y, s) \).

**Remark.** It follows from [OS04] and [Ras03] that \( CFK^\infty(Y, K) \) is independent of the choices of the 2-pointed Heegaard diagram and generic almost complex structure \( J \) in the sense that different choices yield chain homotopy equivalent. From a different perspective, if one equips \( Sym^g(\Sigma) \) with a symplectic form, then the above construction defines the Lagrangian Floer homology of the pair \((\mathbb{T}_\alpha, \mathbb{T}_\beta)\), whose differential counts \( J \)-holomorphic disks in \( Sym^g(\Sigma) \). Gromov started the theory of \( J \)-holomorphic curve [Gro85]. The construction of Floer homology was first provided by Floer [Flo88].

## 2.2 The Definition and Properties of the \( \Upsilon \)–invariant

In this section we generalize the definition of the \( \Upsilon \)–invariant for \( K \) a null-homologous knot in a rational homology sphere based on Livingston’s approach in [Liv17]. We also develop necessary machinery for later discussion related to open book decomposition and contact structure.

### 2.2.1 \( t \)-filtration and \( \Upsilon \)

Fix \( t \in [0, 2] \) and a generator \([x, i, j] \), we start with a real-valued function

\[
 f_t([x, i, j]) = (1 - \frac{t}{2})i + \frac{t}{2}j
\]

on \( CFK^\infty(Y, K, s) = C \). Furthermore, let \( \theta = [x_1, i_1, j_1] + ... + [x_n, i_n, j_n] \) be a chain in \( C \), we also define a function

\[
 F_t(\theta) = \max \{ f_t([x_k, i_k, j_k]) \}.
\]
Proposition 2.2.1. $F_t$ defines a filtration $F^t$ on $C$, where the filtered subcomplexes are given by $F^t_s = f^{-1}_t(-\infty, s]$. Furthermore, $F^t$ is discrete, i.e., for any $s_1 \geq s_2$, $F^t_{s_1}/F^t_{s_2}$ is finite-dimensional.

Proof. Under the boundary map $\partial^\infty(\theta) = \Sigma \partial^\infty[x_k, i_k, j_k]$, where $\partial^\infty$ reduce both $i_k$ and $j_k$. Both $1 - \frac{t}{2}$ and $\frac{t}{2}$ are positive as well so that $F_t(\theta) \geq F_t(\partial^\infty(\theta))$.

For discreteness we see that there are $k_1$ and $k_2$ such that $C(i \leq k_1) \subset F^t_{s_2} \subset F^t_{s_1} \subset C(i \leq k_2)$. Since the algebraic filtration is discrete, so is $F^t$.

Definition 2.2.2. $\nu_t(Y, K, s) = \min \{ F_t(\theta) | \theta is a cycle in C and [\theta] is non-trivial with Maslov grading d(Y, s) \}$.

We can see that $\nu_t(Y, K, s)$ is in fact the minimum $F^t-$filtered level such that the inclusion induced map $H(F_t) \hookrightarrow H(C)$ on homology contains the base element with degree $d(Y, s)$.

Definition 2.2.3. $\Upsilon_{Y, K, s}(t) = -2\nu_t(Y, K, s)$.

When $Y$ is understood from the context, then we drop it from the notation. We say a generator $[x, i, j]$ realizes $\Upsilon_{K, s}(t)$ if $[x, i, j]$ is a summand of a cycle $\theta$ satisfying the condition in definition 2.2.2 and $\nu_t(K, s) = f_t([x, i, j])$.

2.2.2 \( \Upsilon \) as a function of $t$

An initial observation is that $\Upsilon_{K, s}(0) = 0$. Indeed, $f_0([x, i, j]) = i$ is the algebraic filtration.

Theorem 2.2.4. Given $t \in [0, 2]$,

(a) $\Upsilon_{K, s}(t)$ is a continuous piece-wise linear function.
Chapter 2: Generalized $\mathcal{Y}$–Invariant

(b) If $\mathcal{Y}_{K,s}(t)$ is differentiable at $t$, and a generator $[x, i, j]$ realizes $\mathcal{Y}_{K,s}(t)$, then
$$\mathcal{Y}_{K,s}'(t) = i - j = -A(x).$$

(c) $\mathcal{Y}_{K,s}(t)$ is not differentiable at $t$ only if at least two generators $[x, i, j], [x', i', j']$ realize $\mathcal{Y}_{K,s}(t)$.

**Proof.** The proof is essentially the same as [Liv17]. Since $\mathcal{F}^t$ is discrete, for all but finitely many $t$ there is exactly one generator $[x, i, j]$ realizing $\mathcal{Y}_{K}(t)$. For nearby $t$, say $t_1$, $\mathcal{Y}_{K}(t_1)$ is realized by the same generator $[x, i, j]$ so that $\nu_{t_1}(K, s) = (1 - \frac{t_1}{2})i + \frac{t_1}{2}j$. Written differently,
$$\mathcal{Y}_{K,s}(t) = -2\nu_{t}(K, s) = (i - j)t - 2i.$$

Thus $\mathcal{Y}_{K,s}'(t) = i - j$. Furthermore, $\mathcal{Y}_{K,s}(t)$ is not differentiable only if two generators $[x, i, j], [x', i', j']$ realize $\mathcal{Y}_{K,s}(t)$ and $i - j \neq i' - j'$.

**Corollary 2.2.5.** $\mathcal{Y}_{K,s}'(t)$ is between $-g(k)$ and $g(k)$.

**Proof.** The Alexander grading is always between $-g(K)$ and $g(K)$.

**Theorem 2.2.6.** The $\mathcal{Y}$–invariant satisfies the following properties:

(a) $\mathcal{Y}_{Y\#Y', K\#K', s\#s'}(t) = \mathcal{Y}_{Y,K,s}(t) + \mathcal{Y}_{Y',K',s'}(t)$.

(b) $\mathcal{Y}_{Y,K,s}(t) = -\mathcal{Y}_{-Y,K,s}(t)$

(c) $\mathcal{Y}_{K,s}(t) = \mathcal{Y}_{K,s}(2 - t)$.

**Proof.** For part (a), the complex $CFK^\infty(Y\#Y', K\#K', s\#s')$ is bifiltered chain homotopy equivalent to $CFK^\infty(Y, K, s) \otimes CFK^\infty(Y', K', s')$. If $(C, \mathcal{F})$ and $(C', \mathcal{F}')$ are two filtered complexes, there is a natural filtration $\mathcal{F} \otimes \mathcal{F}'$ on $C \otimes C'$:

$$(C \otimes C')_s = \text{Image}(\oplus_{s=s_1+s_2} C_{s_1} \otimes C'_{s_2} \to C \otimes C').$$
It follows from theorem 6.1 in [Liv17] that \( \nu_t \) is additive for each \( t \). Hence
\[
\Upsilon_{Y \# Y', K \# K', s \# s'}(t) = \Upsilon_{Y, K, s}(t) + \Upsilon_{Y', K', s'}(t).
\]

For part (b), the complex \( CFK^\infty(Y, K, s) \) with filtration \( F^t \) has a dual complex \( CFK^\infty(Y, K, s)^* \) with decreasing filtration \( F^{t*} \). \( \nu_t(K) \) can be defined as the maximal filtration level of a class in the dual complex which contains a non-trivial element of cohomology in grading \( d(Y, s) \). Since \( (CFK^\infty(-Y, K, s), F) \cong (CFK^\infty(Y, K, s)^*, -F^*) \). \( \Upsilon_{Y, K, s}(t) = -\Upsilon_{-Y, K, s}(t) \) is proved.

Part (c) follows immediately from switching the role of base points \( w \) and \( z \). ■
Chapter 3

The $\Upsilon$–invariant of Fibered Knots

In this chapter we prove Theorem 1.0.4.

**Theorem 3.0.1.** If $\Upsilon'_{K,s}(t) = -g$ for some $t \in [0,1)$, where $g$ is the genus of the fibered surface $\Sigma$, then $\phi : \Sigma \to \Sigma$ is right-veering. The converse does not hold in general.

We start this chapter by reviewing the definition of right-veering surface diffeomorphism [HKM07].

### 3.1 Right-veering diffeomorphism

Let $\Sigma$ be a compact oriented surface with boundary $\partial \Sigma$, and let $\alpha, \beta : [0,1] \to \Sigma$ be properly embedded oriented arcs with $\alpha(0) = \beta(0) = x \in \partial \Sigma$. Isotope $\alpha$ and $\beta$ so that they intersect transversely with the fewest possible number of intersections. We say that $\beta$ is to the right of $\alpha$ if $(\dot{\beta}(0), \dot{\alpha}(0))$ define the orientation of $\Sigma$ at $x$.

**Definition 3.1.1.** Let $\phi : \Sigma \to \Sigma$ be a diffeomorphism which restricts to the identity map on the boundary $\partial \Sigma$. Let $\alpha$ be a properly embedded oriented arc starting at a
based point $x \in \partial \Sigma$. Then we say $\phi$ is right-veering if for arbitrary based point $x$ and arc $\alpha$, $\phi(\alpha)$ is always to the right of $\alpha$.

### 3.2 Knot Floer homology of fibered knots

Let $K$ be the binding of an open book $(\Sigma, \phi)$ of $Y$ compatible with a contact structure $\xi$. A basis for $\Sigma$ is a collection $\{a_1, ..., a_{2g}\}$ of disjoint, properly embedded arcs in $\Sigma$ whose complement is a disk. Let $b_i$ be an isotopic copy of $a_i$ obtained by shifting the end points of $a_i$ in the direction of $K$ so that $b_i$ intersects $a_i$ at a single point $x_i$. Following [HKM09], we form a pointed Heegaard diagram

$$(S, \beta = (\beta_1, ..., \beta_{2g}), \alpha = (\alpha_1, ..., \alpha_{2g}), w)$$

for $-Y$ by doubling the open book:

- $S = \Sigma \cup -\Sigma$ is the union of two copies of $\Sigma$ glued along the binding $K$,
- $\alpha_i = a_i \cup a_i$,
- $\beta_i = b_i \cup \phi(b_i)$,
- the based point $w$ lies outside of the strip from the isotopies from $a_i$ to $b_i$ as shown in the following figure,

Now we turn the Heegaard diagram into a doubly-pointed Heegaard diagram for $K \subset -Y$. We perform finger moves on the $\beta$ curves in the direction of the orientation of $K$, and place the second based point $z$ inside the region of the isotopies.

The following lemma by Baldwin and Vela-Vick [BVV18] characterize the Alexander grading of generators.
Figure 3.1: the arcs $a_1, a_2$ are red and $b_1, b_2$ are blue. The intersection points $x_1, x_2$ are shown in black dots.

Figure 3.2: A doubly-pointed Heegaard diagram of $K \subset -Y$. The bigon from $y$ to $x$ is shown in grey.

**Lemma 3.2.1.** The Alexander grading of a generator $x$ is the number of components in $-\Sigma \subset S$ minus $g$.

**Proposition 3.2.2.** If $A(x) = -g$, then every component $x$ lies in $\Sigma$, which is an intersection provided by the finger moves.

If $\phi$ is not right-veering, then from [HKM09] there exists a non-separating arc $a_1$ such that $\phi(a_1)$ is to the left of $a_1$. $a_1$ can be completed to a basis $\{a_1, \ldots, a_{2g}\}$.

**Corollary 3.2.3.** Given a generator $x$ with $A(x) = -g$, if $\phi$ is not right-veering, then there is a bigon containing the based point $z$ that connects some other generator $y$ to $x$. Moreover, $A(y) = 1 - g$.

**Proof.** See Figure 3.2. Notice that on $-\Sigma$, $\phi(b_1)$ is to the right of $b_1$. ■
3.3 Proof of Theorem 1.0.4

We will prove the following: if \( \phi : \Sigma \to \Sigma \) is not right-veering and \( \Upsilon_{K,s}(t) = -g \) then \( t \geq 1 \). In fact, we will show that if \( \Upsilon'_{m(K),s}(t) = g \) then \( t \geq 1 \), where \( m(K) \) is the mirror of \( K \). Then the theorem follows from theorem 2.2.6 that \( \Upsilon_{K,s}(t) = -\Upsilon_{m(K),s}(t) \).

Now we consider the complex \( CFK^\infty(-Y,K,s) \) associated to the Heegard diagram compatible with the open book \((\Sigma,\phi)\).

Suppose \( \Upsilon'_{m(K),s}(t_0) = g \) for some \( t_0 \). It follows from Theorem 2.2.4 that \( U^m c \) realizes \( \nu_{t_0}(-Y,K,s) \), where \( c \) is a chain with \( A(c) = -g \). We recall the definition:

**Definition 3.3.1.** \( \nu_t(-Y,K,s) = \min \{ F_t(\theta) | \theta \text{ is a cycle in } C \text{ and } [\theta] \text{ has Maslov grading } d(-Y,s) \} \).

If \( \theta = \sum [x_k, i_k, j_k] \), then \( F_t(\theta) = \max \{ f_t([x_k, i_k, j_k]) \} \).

There exists some cycle \( \eta \in CFK^\infty(-Y,K,s) \) satisfying:

- \( [\eta] \in HF^\infty(-Y,K,s) \) has absolute grading \( d(-Y,s) \).
- \( \eta = U^m c + \eta' \)
- \( \nu_{t_0}(-Y,K,s) = F_{t_0}(\eta') = f_{t_0}(U^m c) = m - \frac{gt_0}{2} \geq F_{t_0}(\eta') \).

Suppose \( F_{t_0}(\eta') = (1 - \frac{t_0}{2})i + \frac{t_0}{2}j \) for some \((i,j)\). Hence,

\[
m - \frac{gt_0}{2} \geq i - \frac{i - j}{2} t_0.
\]

Since \( i - j \leq g \), the inequality holds for any \( 0 \leq t_1 < t_0 \). Thus,

\[
F_{t_1}(\eta) = f_{t_1}(U^m c) = m - \frac{gt_1}{2}
\]
for any $0 \leq t_1 < t_0$. In other words, any summand of $\eta$ other than $U^m c$ can only realize the $\Upsilon$–invariant when $t > t_0$.

If $\phi : \Sigma \to \Sigma$ is not right-veering, then from proposition 3.2.3 there is a generator $y$ such that

- $\partial^{\infty}(U^m y) = U^m c + \theta$, and

- $A(y) = 1 - g$.

![Figure 3.3: This figure shows that we have other generators realizing the $\Upsilon$–invariant for some $1 \leq t \leq t_0$ if there is a bigon from $y$ to $c$.]

Then

$$\partial^{\infty}(\partial^{\infty}U^m y) = \partial^{\infty}U^m c + \partial^{\infty}\theta = 0.$$  

Since $\eta$ is a cycle,

$$\partial^{\infty}\eta = \partial^{\infty}U^m c + \partial^{\infty}\eta' = 0.$$

as well. Therefore, \( \theta + \eta' \) is also a cycle in \( \text{CFK}^\infty(-Y, K, s) \), denoted by \( \delta \). Moreover, \( \delta \) has Maslov grading \( d(-Y, s) \) and

\[
F_{t_0}(\delta) = \max(F_{t_0}(\theta), F_{t_0}(\eta')) \geq F_{t_0}(\eta) = f_{t_0}(U^m c)
\]

because \( F_{t_0}(\eta) = \nu_t(-Y, K, s) = \min \{ F_t(\theta) | \theta \text{ is a cycle in } C \text{ and } [\theta] \text{ has Maslov grading } d(-Y, s) \} \). Thus, \( F_{t_0}(\theta) \geq f_{t_0}(U^m c) \geq F_{t_0}(\eta') \). Suppose \( F_{t_0}(\theta) = (1 - \frac{t_0}{2})i' + \frac{t_0}{2} j' \) for some \((i', j')\). Hence,

\[
m - \frac{gt_0}{2} \leq i' - \frac{i' - j'}{2} t_0.
\]

Again \( i' - j' \leq g \). There is a bigon containing \( z \) from \( y \) to \( c \), so \( y \) and \( c \) are at the same algebraic filtered level. Since \( \partial^\infty(U^m y) = U^m c + \theta \) and \( \partial^\infty \) reduce algebraic filtered level, we conclude that \( m \geq i' \). Therefore, there exists \( t_2 < t_0 \),

\[
m - \frac{gt_2}{2} = i' - \frac{i' - j'}{2} t_2.
\]

Rewrite it as

\[
t_2 = \frac{2(m - i')}{g - (i' - j')}.
\]

Moreover, for some \( t' \in (t_2 - \epsilon, t_2) \), \( \nu_{t'}(-Y, K, s) = F_{t'}(\theta) \) is realized by some generator \([x, i', j']\). Since \( \partial^\infty(U^m y) = U^m c + \theta \) and \( A(y) = 1 - g \),

\[A(x) = j' - i' \geq 2 - g\]

and

\[m - i' \geq j' - i' - (1 - g)\]

Therefore,

\[
t_2 \geq \frac{2(j' - i' - (1 - g))}{g - (i' - j')} = 2 - \frac{2}{g - (i' - j')} \geq 1.
\]

and \( t_0 \geq t_2 \geq 1 \) as desired. \( \blacksquare \)
3.4 Examples

Example 3.4.1. For fibered knots $K \subset S^3$ with less than 10 crossings, $\Upsilon'_K(t) = -g$ for some $t \in [0,1)$ if and only if $K$ supports the unique tight contact structure on $S^3$.

Proof. For any knot in $S^3$, Ozsváth and Szabó [OSS17] prove that $\Upsilon_K(t) = -\tau(K)t$ for small $t$. Moreover, if the fibered knot supports the unique tight contact structure on $S^3$, then $\tau(K) = g(K)$.

On the other hand, a fibered knot $K$ supports a tight contact structure in $S^3$ if and only if it is strongly quasi-positive. We look up the monodromies of fibered knots under 10 crossings that are not strongly quasi-positive from [Kno]. By brute force we find that none of them are is right-veering. Therefore, by theorem 1.0.4, $\Upsilon'_K(t) > -g$.

Example 3.4.2. The converse of theorem 1.0.4 does not hold even for fibered knots in $S^3$.

Proof. Let us consider the knot $K = 8_{20}$, which is a slice and fibered knot. The $(p,1)-$cable $K_{p,1}$ is also slice and fibered. Indeed, $8_{20}$ is the pretzel $P(3,-3,2)$. One can construct a slice disks by adding two 1-handles and three 2-handles in $B^4$. The slice disk of the $(p,1)-$cable can be obtained by stacking $p$ copies of the disks constructed above and connecting them with half-twisted bands. Therefore, $\Upsilon_{K_{p,1}}(t) = 0$.

On the other hand, Kazez and Roberts [KR12] show that the fractional Dehn twist coefficient of a fibered knot obtained by cabling is $\frac{1}{p} > 0$. Hence the monodromy of $K$ is right-veering.
Bibliography


[Kno] https://www.indiana.edu/ knotinfo/.


