Contributing or Free-Riding? Voluntary Participation in a Public Good Economy

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Boston College Working Papers in Economics, 2008

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August 4, 2010

Abstract

We consider a (pure) public goods provision problem with voluntary participation in a quasi-linear economy. We propose a new hybrid solution concept, the *free-riding-proof core (FRP-Core)*, which endogenously determines a contribution group, public goods provision level, and how to share the provision costs. The FRP-Core is always nonempty in public goods economies but does not usually achieve global efficiency. The FRP-Core has support from both cooperative and noncooperative games. In particular, it is equivalent to the set of perfectly coalition-proof Nash equilibria (Bernheim, Peleg, and Whinston, 1987) of a dynamic game with players' participation decisions followed by a common agency game of public goods provision. We illustrate various properties of the FRP-Core with an example. We also show that the equilibrium level of public goods shrinks to zero as the economy is replicated.

Keywords endogenous coalition formation, externalities, public good, perfectly coalition-proof Nash equilibrium, free-riders, free-riding-proof core, lobbying, common agency game

*The paper is an expanded version of "Contributing or Free-Riding? A Theory of Endogenous Lobby Formation," by the same authors. We are grateful to Francis Bloch, Bhaskar Dutta, Matt Jackson, Hubert Kempf, Michihiro Kandori, Chiu Yu Ko, Rachel Kranton, Michel Le Breton, Akihiko Matsui, Herve Moulin, Elena Paltseva, Ryusuke Shinohara, Sang-Seung Yi, and the participants of many seminars and workshops/conferences for their comments. Konishi thanks Kyoto Institute of Economic Research for providing an excellent research environment. Both authors are grateful to two anonymous referees and the editor Ed Green for helpful comments and suggestions. All remaining errors are, of course, our own.

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1 Introduction

The free-riding problem is a central issue in collective decision-making. Examples include lobbies that are formed to seek a government's protection. Firms in an industry may form a lobby to influence the government's trade and industrial policies; but as long as a sufficient number of firms participates in the lobbying (so that the resulting protection level is reasonably high), some firms may want to stay out and free-ride on others. Free-riding incentives also exist in other examples of collective decision making, such as firms' cartel formation and international agreements to tackle the problem of global climate change. The effectiveness of such lobbies, cartels, and international agreements depends on the composition of the active participants in these activities, which is heavily influenced by free-riding incentives.

These problems can be regarded as pure public goods provision problems with voluntary participation. Players are faced with a choice between actively participating in public goods provision and free-riding on the contributors; if a player participates (and only in such cases), she needs to share the cost of public goods provision, although she can then influence the level of provision. A notable feature of the public goods provision problem is that the expansion of the contribution group is always beneficial to everybody. Since benefits from public goods will be extended beyond the contribution group members, however, there always exist free-riding incentives, which generally grow as the contribution group expands. Therefore, it is important to consider potential coalitional deviations that are immune to further deviations. A proposal of membership expansion, which is regarded as a coalitional deviation, may not be immune to free-riding by the incumbents, however, so some of these Pareto-improving proposals may not be credible. To define an appropriate solution to the public goods provision problem with voluntary participation, therefore, we must consider all possible coalitional deviations that are immune to further deviations.

In this paper, employing a quasi-linear economy, or equivalently a transferable utility (TU) framework, we propose the *free-riding-proof core (FRP-Core)* for the public goods provision problem with voluntary participation, which is an institution-free solution concept with farsightedness. The FRP-Core is a hybrid solution concept, as it is required to be immune not only to coalitional deviations to create an alternative contribution group (in line with cooperative game theory) but to unilateral free-riding deviations (in line with non-cooperative game theory); the FRP-Core can be considered as the core without binding agreements. It determines the formation of a contribution group, public goods provision level, and a payoff allocation within the group.

The FRP-Core is defined in the following way. First, for every possible contribution group, we collect all allocations such that (i) they are immune to all coalitional deviations by subsets of the group to reorganize the contribution group, and (ii) no member of the group is better off by unilaterally opting out of the group to free-ride. These allocations constitute the set of internally stable allocations for the contribution group. Second, for each contribution group, we collect all internally stable allocations that are not blocked by any other contribution group's deviations with their internally stable allocations. The free-riding-proof core (FRP-Core) is the union of such stable sets over all possible contribution groups. In a pure public goods economy, the FRP-Core is always nonempty (Proposition 2).

The FRP-Core has not only intuitive appeal, but also useful correspondences with cooperative and noncooperative game solution (equilibrium) concepts. On one hand, the FRP-Core is equivalent to the core when the set of feasible allocations is restricted to those of the free-riding-proof (Theorem 1). Thus, the FRP-Core is a natural and appealing solution concept from the viewpoint of cooperative game theory, which provides solutions that are robust to changes in detailed specifications of discussed situations. On the other hand, the FRP-Core is equivalent to the set of equilibrium outcomes of a simple extensive-form game such that players individually decide whether or not to participate in the public goods contribution group in the first stage, followed by a common agency game (Bernheim and Whinston, 1986) of public goods provision, with players in the contribution group as principals and the government as the agent. This equivalence provides support for the FRP-Core from the viewpoint of noncooperative game theory.

The equivalence result requires further explanation. The common agency game introduced by Bernheim and Whinston (1986) is a game in which players (principals) simultaneously offer their individual contribution schedules to the agent to try to affect the agent's action in their own interests.[1] To refine the Nash equilibrium, which tends to yield a large equilibrium set due to the coordination problem, Bernheim and Whinston (1986) propose the coalition-proof Nash equilibrium (CPNE), a communication-based equilibrium concept for the common agency game. In the first stage of our extensive-form game, players individually decide whether or not to participate in the contribution group, which also involves the coordination problem. Therefore, it is natural for us to adopt a dynamic extension of CPNE, which is the perfectly coalition-proof Nash equilibrium (PCPNE; Bernheim, Peleg, and Whinston, 1987). The equilibrium concept PCPNE fits particularly well with the public goods provision problem in which players may communicate with each other as to whether or not they join the contribution group.

The equilibrium concept PCPNE has a few merits: (i) it determines a contribution group, public goods provision level, and how to share the provision costs all together; (ii) it allows players to propose a (coalitional) deviation plan in which they coordinate their strategies (including the ones in subgames) through communications; and (iii) it requires credibility of proposed deviation plans so that no credible deviation remains in equilibrium. CPNE and PCPNE are strategy profiles that are immune to (recursively defined) credible group deviations with their strategies coordinated. A credible deviation is a deviation that is immune to further nested credible deviations. The credibility requirement enables us to exclude noncredible proposals of membership expansion from consideration. We show the equivalence between the FRP-Core and PCPNEs of the aforementioned extensive-form game of public goods provision (Theorem 2). In contrast, Appendix C examines some other natural candidates for the equilibrium concept, such as the subgame perfect Nash equilibrium, and some other forms of the public goods provision game, and shows that these other equilibrium allocations do not coincide with the FRP-Core allocations. What distinguishes PCPNE from other solution concepts is property (ii) above. Players in a deviation group can discuss which (credible) strategies they take in the following subgames, which eliminates ungrounded fears

[1] This game has been widely applied to political economy models with lobbying, especially in the field of international trade (e.g., Grossman and Helpman, 1994).

about possibly unfavorable consequences of the deviation and hence solves the coordination problem.

We examine properties of the set of FRP-Core allocations with a simple example in which players differ only in their willingness-to-pay for a public good, and show that (i) there are multiple possible equilibrium contribution groups in general, (ii) an equilibrium contribution group may not include the player with the highest willingness-to-pay, and (iii) equilibrium contribution-group members may not be consecutive in their willingness-to-pay, i.e., there may be an outsider whose willingness-to-pay lies between those of two contributors.

We also analyze how equilibrium public goods provision changes as the economy grows in size. Following Milleron's (1972) notion of replicating a public goods economy, we prove that the equilibrium public goods provision level converges to zero as the economy grows (Theorem 3).[2]

This paper is organized as follows. The next two subsections briefly discuss some related literature. Section 2 sets out our public goods provision game and introduces the FRP-Core as a solution concept. We also provide a simple characterization of the FRP-Core (Theorem 1), which indicates that the FRP-Core is a natural solution concept for the public goods provision problem with voluntary participation. In Section 3, we provide a noncooperative voluntary participation game and propose PCPNE as an equilibrium concept. In Section 4, we prove the equivalence between PCPNE and the FRP-Core (Theorem 2). In Section 5, we provide an example to reveal some interesting properties of the FRP-Core. Section 6 considers a replica economy and shows that the public goods provision level shrinks to zero as the economy is replicated in a certain way (Theorem 3). In Section 7, we conclude with a discussion of the robustness of our results to the utility specification. In particular, we argue that all the results would be preserved with some additional mild assumptions even if we adopt Gorman-form utilities (Bergstrom and Cornes, 1983) instead of quasi-linear utilities. Appendix A provides useful properties of the core of convex games and an algorithm that

[2]Muench (1972), Milleron (1972), and Conley (1994) discuss the difficulty of replicating a public goods economy and offer various possible methods. Milleron's notion of replication is to split endowments with replicates and adjust preferences so that agents' concerns for the private good are relative to the size of their endowments. This notion is employed by Healy (2007).

finds a core allocation starting with an arbitrary utility vector. Appendix B collects proofs of our results. Appendix C examines some other equilibrium concepts and extensive-form games to be compared with the PCPNE of our extensive-form game (or the FRP-Core).

1.1 Related Literature in the Theory of Coalition Formation

Since the public good in our problem is pure, so that outsiders can enjoy the benefits from public goods provision, our problem belongs to the class of coalition formation problems with spillovers (externalities). The literature on this subject is very large (see Bloch 1997, and Ray 2007 for overviews); here we discuss only the most closely related papers to ours.

CPNE has been adopted as an equilibrium concept in the theory of coalition formation. Thoron (1998) examines the formation of a single cartel and shows that the coalition-proof stable cartel is uniquely determined adopting the CPNE as a solution concept. Yi and Shin (2000) investigate a joint research venture allowing the formation of multiple R&D cartels; they examine the existence of a CPNE and its properties. In these papers, credible coalitional deviations are considered only at the stage of players' decisions as to whether or not to participate in a coalition. In the extensive-form game of this paper, we require both participation and contribution decisions of players to be immune to credible coalitional deviations. To our knowledge, this paper is the first to use PCPNE as a solution concept of the coalition formation problem, which endogenously determines the coalition formation and the allocation of payoffs within the coalition.

The literature also includes studies on coalition formation regarding the public goods provision. Assuming identical players and nontransferable payoffs, Bloch (1997) provides a complete comparison of the equilibrium coalition structures, employing various solution concepts and game forms. Also assuming identical players, Ray and Vohra (2001) characterize the equilibrium coalition structure and payoff allocations for a standard sequential coalitional bargaining game of public goods provision. These authors allow multiple coalitions to form for public goods provision. In contrast, we assume that there is only one coalition and its members are the only ones who provide the public good.[3] But, we allow players to be

[3]With our second-stage common agency game, this difference turns out to be unimportant. Even if

heterogeneous, and more importantly provide a simple characterization of the set of PCPNE allocations utilizing its equivalence with the FRP-Core.

Last but not least, there is the literature on noncooperative coalition bargaining games that discusses the relationship between the equilibrium outcome(s) in the limit cases and the core. Chatterjee, Dutta, Ray, and Sengupta (1993) show that the Markov equilibrium outcome is a subsolution of the core (in convex TU games). Perry and Reny (1994) and Moldovanu and Winter (1995) provide coalitional bargaining games that implement the core.[4] The latter two papers are particularly related to our paper, since the set of outcomes of CPNE is equivalent to the core in our common agency game.[5] In Appendix C, we compare the set of PCPNEs of our game with equilibrium outcomes of these noncooperative bargaining games preceded by a voluntary participation game.

1.2 Related Literature on Voluntary Participation Mechanisms in a Public Goods Economy

It is well known that public goods provision is subject to free-riding incentives. Although Samuelson's (1954) view of this problem was pessimistic, Groves and Ledyard (1977) show that efficient public goods provision can be achieved in Nash equilibrium if individual rationality is not required. Hurwicz (1979) and Walker (1981) show that the Lindahl mechanism is implementable. Subsequently, numerous mechanisms have been proposed to improve the properties of mechanisms. They all assume, however, that players have no freedom to make participation decisions about the mechanism, i.e., players' participation in the mechanism is always assumed.

Introducing outside opportunities by a "reversion function" (such that each outcome is mapped to another outcome in the case of no participation), Jackson and Palfrey (2001) analyze the implementation problem when players' participation in a mechanism is voluntary.

multiple coalitions are formed, the equilibrium public goods provision level will be the same as when the union of them is the contribution group. We need to assume, however, that nonparticipants cannot contribute in the second stage.

[4]We thank a referee for bringing our attention to this literature.

[5]Laussel and Le Breton (2001) show that the set of CPNE outcomes is equivalent to the core when the underlying TU game of a common agency game is convex. Convexity of the TU game is satisfied in the common agency game in a public goods provision problem.

Although their reversion function is very general, it assigns the same outcome no matter who deviates from the original outcome. Thus, the method may not be suitable for a public goods provision problem in which different players' deviations from participation may generate different outcomes. Taking this consideration into account, Healy (2007) analyzes the implementation problem in a public goods economy demanding all players' participation in equilibrium (i.e., equilibrium participation). He shows that as the economy is replicated in Milleron's (1972) sense, the set of equilibrium allocations of any mechanism that satisfies the equilibrium participation condition converges to the endowment. Although the equilibrium public goods provision level also converges to zero as the economy is replicated in our model, we allow players not to participate in the contribution group in equilibrium, and it is indeed a source of underprovision of the public good, unlike Healy's (2007).

Closest to our noncooperative framework is the one by Saijo and Yamato (1999), who are the first to consider a two-stage voluntary participation game of a public goods economy, without requiring all players' participation in equilibrium.[6] They show a negative result on the efficiency of public goods provision, and then characterize subgame perfect equilibria in the case of symmetric Cobb-Douglas utility. In contrast, we show that the set of PCPNE of a common agency game with a participation decision is equivalent to our FRP-Core, allowing heterogeneous players that have quasi-linear utility functions.

Palfrey and Rosenthal (1984) show that in a binary public goods provision game where symmetric players voluntarily make participation decisions, all pure-strategy Nash equilibria are efficient (if contributions are not refundable in the case of no provision). With asymmetric players, there are many Nash equilibria with different levels of cooperation. Shinohara (2009) examines a public goods provision problem with decreasing marginal benefits, and shows in the case of homogeneous players that it becomes harder to support efficient allocations as the efficient level of the public good rises and hence the number of participants needed to

[6]In the field of international trade, Bombardini (2008) and Paltseva (2006) extend Grossman and Helpman's (1994) menu-auction political-economy model to incorporate firms' voluntary participation in lobbies; they consider the cases in which firms in oligopolistic, import-competing industries make participation decisions. Unlike our noncooperative game framework, Bombardini (2008) considers only the most efficient contribution group, while Paltseva (2006) assumes that firms are symmetric and derives subgame perfect equilibrium as opposed to PCPNE.

provide the public good increases. Our Theorem 3 is somewhat similar to this result.

2 The Model

This section sets out the contribution game in which all players' interests accord with each other, although the intensity of their interests can be heterogeneous. We first describe the problem of interest and then propose the FRP-Core as an appealing solution to the problem.

2.1 Public Goods Provision Problem

A stylized public goods model is defined as follows. There is a public good whose provision level is denoted by $a \in A = \mathbb{R}_+$.[7] Provision cost function $C : A \to \mathbb{R}_+$ is continuous and strictly increasing with $C(0) = 0$. The government provides the public good, and its cost is regarded as the government's disutility from the provision; the government's utility from providing a units of the public good is $v_G(a) = -C(a)$. Player i's utility function is quasi-linear such that her utility from the consumption of the public good is $v_i(a)$ and the net consumption x of the private goods enters the function linearly, i.e., $v_i(a) + x$, where $v_i : A \to \mathbb{R}_+$ is strictly increasing with $v_i(0) = 0$. In order to guarantee the existence of a nontrivial solution, we assume that (i) there exists $\tilde{a} \in A$ such that $v_i(\tilde{a}) - C(\tilde{a}) > 0$ for all $i \in N$, where N is the set of n players, and (ii) there is $\hat{a} \in A$ such that $\sum_{i \in N} v_i(a) - C(a) < 0$ for all $a > \hat{a}$. The only new element in this standard public goods provision game is that every player has a choice as to whether or not to participate in the contribution group. The contribution group, therefore, may be a proper subset of all players.

2.2 Free-Riding-Proof Core

We first define an intuitive hybrid solution concept, the *free-riding-proof core (FRP-Core).* In short, the FRP-Core is the Pareto-optimal set of the Foley-core allocations of all contribution groups that are immune to free-riding incentives. The FRP-Core will be shown to be always

[7]For our equivalence results (Theorems 1 and 2), we only need comonotonic preferences over an abstract agenda set A. The extension is straightforward. We focus on the one-dimensional public goods economy just for simplicity.

nonempty in the public goods provision problem.

Our interests in the public goods provision problem are twofold: (i) which group provides how much of the public goods, and (ii) how the benefits and costs of the public good are shared by group members. For $S \subseteq N$ with $S \neq \emptyset$, we define

$$V(S) \equiv \max_{a \in A} \left[\sum_{i \in S} v_i(a) - C(a) \right],$$

and

$$a^*(S) \equiv \arg\max_{a \in A} \left[\sum_{i \in S} v_i(a) - C(a) \right].$$

An *allocation for* S is (S, a, u) such that $u \in \mathbb{R}^n_+$, $\sum_{i \in S} u_i \leq \sum_{i \in S} v_i(a) - C(a)$, and $u_j = v_j(a)$ for all $j \notin S$. An *efficient allocation for* S is an allocation (S, a, u) such that $\sum_{i \in S} u_i = V(S)$ with $a \in a^*(S)$. (Henceforth, we assume that $a^*(S)$ is a singleton just for simplicity.) Note that $N \backslash S$ are passive free-riders, and they do not contribute at all. Let $X(S)$ be the collection of all efficient allocations for S. A *free-riding-proof efficient allocation for* S is an efficient allocation for S, $(S, a^*(S), u) \in X(S)$ such that

$$u_i \geq v_i(a^*(S \backslash \{i\})) \text{ for any } i \in S.$$

That is, under a free-riding-proof efficient allocation, no player in S has an incentive to opt out of the contribution group while enjoying the public good provided by the remaining players in S. Let $X^{FRP}(S)$ be the set of all free-riding-proof efficient allocations for S. It should be emphasized that $X^{FRP}(S)$ can be empty when S is a large set; with a large number of members, it becomes harder to satisfy free-riding-proofness.

Given that S is the contribution group, a natural way to allocate utility among the members is to use the core (Foley, 1970).[8] Focusing on coalition S and its subsets, we write $Core(S) = \left\{ (S, a^*(S), u) \in X(S) : \sum_{i \in T} u_i \geq V(T), \ \forall T \subseteq S \right\}$, the set of all core allocations

[8] The Foley core is the standard core concept of a public goods economy. In the definition of the Foley core, when a subset of the contribution group decides to deviate from an allocation, the original agreement on the level of public goods provision and the cost-sharing are totally abandoned and the public good is provided solely by the deviating coalition. Others, including the members (if any) of the original contribution group in question who are not in the deviating coalition, are still able to enjoy the public good. In the Foley notion of the core, what is important is whether or not members of a blocking coalition would be better off by the deviation, while well-being of other players is not the issue.

for S, which is immune to all subcoalitions' group deviations. Obviously, a core allocation for S may not be immune to free-riding incentives for the members of S. Thus, we define the *FRP-core allocation for S* by $Core^{FRP}(S) = Core(S) \cap X^{FRP}(S)$. An FRP-Core allocation for S is a core allocation for S that is immune to unilateral free-riding deviations, and $Core^{FRP}(S)$ is the set of all FRP-Core allocations for S.[9] The set $Core^{FRP}(S)$ is a collection of *internally stable* allocations for S in the sense that no subgroup of S has an incentive to deviate to form an alternative contribution group, and no player in S has an incentive to free-ride. Similarly to $X^{FRP}(S)$, $Core^{FRP}(S)$ may be empty for a large group S, but it is nonempty for small groups (for singleton groups in particular).

Now, we consider allocations that are "fully" stable against any coalitional blocking. A coalition T *(weakly) blocks* an allocation $(S, a^*(S), u)$ via an allocation $(T, a^*(T), u')$ if and only if (i) $u'_i \geq u_i$ for all $i \in T$ and $u'_j > u_j$ for at least one member $j \in T$, (ii) $\sum_{i \in T} u'_i = V(T)$, and (iii) $\sum_{i \in T'} u'_i \geq V(T')$ for all $T' \subset T$.[10] The solution concept free-riding-proof core (FRP-Core) is a collection of all FRP-Core allocations for all S, which are not blocked by any coalitions T with an FRP-Core allocation for T. That is,

$$\begin{aligned} Core^{FRP} &= \left\{ (S, a^*(S), u) \in \cup_{S' \subseteq N} Core^{FRP}(S') : \text{ there does not exist } T \right. \\ &\qquad \left. \text{and } (T, a^*(T), u') \in Core^{FRP}(T) \text{ that (weakly) blocks } (S, a^*(S), u) \right\}. \end{aligned}$$

The FRP-Core is a collection of internally stable allocations for some coalition that are not blocked by any other coalition with an internally stable allocation; we impose a credibility constraint for legitimate coalitional deviations, regarding non-internally-stable coalitional deviations as noncredible because there would be further deviations from such deviations.

The above hybrid solution concept is natural and appealing, but it might not appear to be easy to work with when applied to specific problems. For example, it is not immediately

[9] The FRP-core allocations would not be affected even if we allow free-riding deviations by multiple players in our public goods provision game. If an efficient allocation is immune to any unilateral free-riding deviation, it is also immune to any free-riding deviations by multiple players, since the reduction of the public good would become larger if more players free-ride, so that every player prefers free-riding by herself if she free-rides at all.

[10] We use "weak" blocking and weak Pareto-domination in this paper, partly because it will give us clearer results in the noncooperative game we consider in the next section.

clear if FRP-Core is nonempty. The following proposition provides a useful property of the core for subsets of N.

Proposition 1. If a core allocation for T, $(T, a^*(T), u') \in Core(T)$, blocks $(S, a^*(S), u) \in Core(S)$, then $a^*(S) < a^*(T)$ and $(T, a^*(T), u') \in Core(T)$ (weakly) Pareto-dominates $(S, a^*(S), u) \in Core(S)$.

Proposition 1 claims that if a core allocation for T is preferred to a core allocation for S by all members of T, then the former allocation Pareto-dominates the latter in our pure public goods economy. This proposition simplifies the characterization of the FRP-Core, allowing us to rewrite it as

$$Core^{FRP} = \overline{\cup_{S' \in 2^N} Core^{FRP}(S')},$$

where $\overline{\Omega}$, for an arbitrary set of allocations Ω, denotes the Pareto frontier of Ω (i.e., an allocation (S, a, u) is in a set $\overline{\Omega}$ if and only if there is no allocation $(T, a', u') \in \Omega$ such that $u' \geq u$ with $u'_i > u_i$ for some $i \in N$). This property assures that $Core^{FRP}$ is nonempty.

Proposition 2. $Core^{FRP} = \overline{\cup_{S' \in 2^N} Core^{FRP}(S')} \neq \emptyset$.

We can also characterize $Core^{FRP}$, using the standard core concept on a restricted allocation set. Take the union of all free-riding-proof efficient allocations for S over all $S \subseteq N$ to obtain $X^{FRP} = \cup_{S \subseteq N} X^{FRP}(S)$. Then we apply the core concept to the set of free-riding-proof allocations X^{FRP}:

$$\begin{aligned} Core \left\langle X^{FRP} \right\rangle &= \big\{ (S, a^*(S), u) \in X^{FRP} : \text{ there does not exist } (T, a^*(T), u') \in X^{FRP} \\ &\quad \text{such that } u'_i \geq u_i \text{ for all } i \in T, \text{ and } u'_j > u_j \text{ for some } j \in T \big\}. \end{aligned}$$

We have the following theorem.

Theorem 1. $Core^{FRP} = Core \left\langle X^{FRP} \right\rangle$.

Two remarks follow. First, for the standard transferable utility (TU) and non-transferable utility (NTU) characteristic-function-form game, Ray (1989) defines credible coalitions recursively on nested coalitions and defines the credible core; he shows that the core and the credible core are equivalent. Despite the facts that our game has externalities due to spillovers of the public good and that the grand coalition usually does not support the FRP-Core, his argument extends to our case such that the credible core of X^{FRP} coincides with $Core^{FRP}$. Second, we impose efficiency on the allocations in the definition of X^{FRP} only for simplicity. We can easily allow inefficient allocations in the definition of feasible allocations by using $\tilde{X}^{FRP}(S) = \{(S, a^*(S), u) : \sum_{i \in S} u_i \leq V(S)$, and $u_i \geq v_i(a^*(S \backslash \{i\}))$ for all $i \in S\}$ and letting $\tilde{X}^{FRP} = \cup_{S \subseteq N} \tilde{X}^{FRP}(S)$. However, we would still require each contribution group S to provide the public good at the efficient level, i.e., $a(S) = a^*(S)$. We need to assign a public goods provision level to every $S \subset N$ in order to define the free-riding-proof allocations; the efficient public goods provision level $a^*(S)$ is a natural candidate to be assigned to each group S.

3 A Voluntary Participation Game

We discuss the endogenous contribution-group formation and its consequences on public goods provision. We first define the extensive-form, public-goods contribution game with voluntary participation; the first stage is a group formation game, followed by the second-stage common agency game played by group S that has been formed in the first stage. For this extensive-form game, we require not only that the common-agency stage of public goods provision be coalition-proof but also that the contribution-group formation itself be coalition-proof. As an extension of CPNE for strategic-form games to the one for extensive-form games, Bernheim, Peleg, and Whinston (1987) define the perfectly coalition-proof Nash equilibrium (PCPNE) as the coalition-proof Nash equilibrium for multi-stage games.

The first-stage *group-formation game* is such that each player $i \in N$ chooses her action from the set $\Sigma_i^1 = \{0, 1\}$, where 0 and 1 represent non-participation and participation, respectively, i.e., player i announces her participation decision. Once action profile $\sigma^1 = (\sigma_1^1, ..., \sigma_n^1) \in \Sigma^1 = \Pi_{j \in N} \Sigma_j^1$ is selected, then the contribution game takes place in the second stage with the set of active players $S(\sigma^1) = \{i \in N : \sigma_i^1 = 1\}$. Since the agent's (the government's) choice in the third stage is a mechanical decision problem, we incorporate this stage in the second-stage contribution game (following Bernheim and Whinston, 1986).

The second-stage game is a common agency game played by participating principals $S(\sigma^1)$, as analyzed by Bernheim and Whinston (1986). The set $N \backslash S(\sigma^1)$ is the set of passive free-riders. Each player $i \in S(\sigma^1)$ simultaneously offers a contribution schedule $\tau_i : A \to \mathbb{R}_+$. Given the profile of contribution schedules $\tau_{S(\sigma^1)} = (\tau_i(a))_{i \in S(\sigma^1)}$, the government G (the agent) chooses a public goods provision level $a \in A$ that maximizes its net payoff:

$$\begin{aligned} u_G(a, \tau_{S(\sigma^1)}) &= \sum_{i \in S(\sigma^1)} \tau_i(a) + v_G(a) \\ &= \sum_{i \in S(\sigma^1)} \tau_i(a) - C(a), \end{aligned}$$

where the first term on the right-hand side of the last equation is the total contribution and the second term is the cost of public goods provision. If the government chooses $a \in A$, then player i obtains her payoff

$$u_i(a, \tau_i(a)) = v_i(a) - \tau_i(a),$$

for $i \in S(\sigma^1)$, and

$$u_i(a) = v_i(a),$$

for $i \notin S(\sigma^1)$. The government's optimal choice is described by

$$a^*(S, \tau_{S(\sigma^1)}) \in \arg\max_{a \in A} u_G(a, \tau_{S(\sigma^1)}),$$

with a slight abuse of notation. Let $\mathcal{T}$ be the set of all contribution schedules $\tau_i : A \to \mathbb{R}_+$. Player i's second-stage strategy σ_i^2 is a mapping $\sigma_i^2 : 2^N \backslash \{\emptyset\} \to \mathcal{T}$: i.e., a contribution schedule is assigned to each subgame. Note that in subgame $S \in 2^N \backslash \{\emptyset\}$ where $i \notin S$, $\sigma_i^2(S) : A \to \mathbb{R}_+$ is irrelevant to the outcome. Nevertheless, we include non-participant's second-stage strategies for notational simplicity. The set of player i's second-stage strategies is denoted by Σ_i^2.

3.1 Perfectly Coalition-Proof Nash Equilibrium for the Contribution-Group Participation Game

Following Bernheim, Peleg, and Whinston (1987), we define PCPNE for our two-stage game. Let $\Sigma \equiv \Pi_{i \in N} \Sigma_i$. Player i's strategy is $\sigma_i = (\sigma_i^1, \sigma_i^2) \in \Sigma_i = \Sigma_i^1 \times \Sigma_i^2$ and her payoff function is $u_i : \Sigma \to \mathbb{R}$ as described above.

For $T \subseteq N$, we consider a *reduced game* $\Gamma(T, \sigma_{-T})$ in which only players in T are active while players in $N \setminus T$ are passive such that they always choose σ_{-T}. We also consider *proper subgames* for every $\sigma^1 \in \Sigma^1$, and *reduced subgames* $\Gamma(T, \sigma^1, \sigma^2_{-T})$ in a similar way. A *perfectly coalition-proof Nash equilibrium (PCPNE)* $(\sigma^*, a^*) \equiv ((\sigma_i^{1*}, \sigma_i^{2*})_{i\in N}, a^*)$ is defined recursively as follows.

Definition. (Bernheim, Peleg, and Whinston, 1987)

(i) In a single-player, single-stage subgame $\Gamma(\{i\}, \sigma^1, \sigma^2_{-\{i\}})$, strategy $\sigma_i^{2*}(S(\sigma^1)) \in \mathcal{T}$ and agenda a^* chosen by the agent is *PCPNE* if σ_i^{2*} maximizes u_i through the choice of a^*.

(ii) Let (n, t) be the pair of the number of players and the number of stages of the reduced (sub-)game, where $t \in \{1, 2\}$. Let $(n, t) \neq (1, 1)$. Assume that PCPNE has been defined for all games with m players and r stages, where $(m, r) \leq (n, t)$ with $(m, r) \neq (n, t)$.

(a) For any game Γ with n players and t stages, $(\sigma^*, a^*) \in \Sigma \times A$ is *perfectly self-enforcing*, if for all proper subset T of the n players, (σ_T^*, a^*) is PCPNE in the reduced game $\Gamma(T, \sigma^*_{-T})$, and if the restriction of σ^* to any proper subgame forms a (P)CPNE in that subgame,

and

(b) for any game Γ with n players and t stages, (σ^*, a^*) is a PCPNE if it is perfectly self-enforcing, and if there does not exist another perfectly self-enforcing pair $(\sigma, a) \in \Sigma \times A$ such that $u_i\,(a, \sigma_i) \geq u_i\,(a^* \sigma_i^*)$ for all $i = 1, ..., n$ with at least one strict inequality.[11]

[11] Bernheim, Peleg, and Whinston (1987) define the PCPNE based on strictly improving coalitional deviations. We adopt, however, a definition based on weakly improving coalitional deviations, since the theorem on menu auction in Bernheim and Whinston (1986), to which we appeal, uses CPNE based on weakly improving deviation. For details on these two definitions, see Konishi, Le Breton, and Weber (1999).

For any $T \subseteq N$ and any strategy profile σ, let $PCPNE(\Gamma(T, \sigma_{-T}))$ denote the set of PCPNE strategy profiles for T in the reduced game $\Gamma(T, \sigma_{-T})$. A strategic coalitional deviation (T, σ'_T, a') from any strategy profile (σ, a) is *credible* if $(\sigma'_T, a') \in PCPNE(\Gamma(T, \sigma_{-T}))$. A PCPNE is a strategy profile that is immune to any credible coalitional deviation. An *outcome allocation* for (σ^*, a^*) is a list $(S(\sigma^{1*}), a^*, u, u_G) \in 2^N \times A \times \mathbb{R}^N \times \mathbb{R}$, where (u, u_G) is the resulting utility allocation for players and the government.

There are two remarks to be made about PCPNE.

First, if a coalition T wants to deviate in the first stage, it can orchestrate the whole plan of the deviation by assigning a new CPNE to each subgame so that the target allocation (by the deviation) would be attained as a PCPNE of the reduced game $\Gamma(T, \sigma_{-T})$.

Second, the definition of PCPNE coincides with the *coalition-proof Nash equilibrium* (CPNE) in the (static) second stage. Thus, a CPNE needs to be assigned to each subgame. There are useful characterizations of CPNE of a common agency game in the literature. The first characterization is provided by Bernheim and Whinston (1986). Consider a subgame S, and denote player i's strategy in this subgame $\sigma_i^2(S) : A \to \mathbb{R}_+$ by $\tau_i : A \to \mathbb{R}_+$. They introduce a concept, called truthful strategies, where τ_i is *truthful relative to* $\bar{a}$ if and only if for all $a \in A$ either $v_i(a) - \tau_i(a) = v_i(\bar{a}) - \tau_i(\bar{a})$, or $v_i(a) - \tau_i(a) < v_i(\bar{a}) - \tau_i(\bar{a})$ with $\tau_i(a) = 0$. A *truthful Nash equilibrium* (τ_S^*, a^*) is a Nash equilibrium such that τ_i^* is truthful relative to $a^* \in A$ for all $i \in S$. Bernheim and Whinston (1986) show that (i) every truthful Nash equilibrium is a CPNE, and (ii) the set of truthful equilibria and that of CPNE in the utility space are equivalent, and they provide a useful characterization of CPNE in the utility space. Laussel and Le Breton (2001) further analyze CPNE in utility space. One of their results provides a nice characterization of CPNE when payoff functions satisfy a special (yet useful) property, the *comonotonic payoff property*: $u_i(a) \geq u_i(a')$ if and only if $u_j(a) \geq u_j(a')$ for all $i, j \in S$ and all $a, a' \in A$. Obviously, this property is satisfied in our public goods provision problem.

Fact. (Laussel and Le Breton, 2001) Consider a common agency problem $\Gamma = (S, A, (\mathcal{T}, v_i)_{i \in S}, C)$ played by the set S of the principals and the agent G with a comonotonic payoff property.

Then, in all CPNEs of the common agency game, agent G obtains $u_G = \max_{a \in A}[-C(a)]$ (no-rent property), and the set of CPNE in utility space is equivalent to the core of the characteristic function game $(\tilde{V}(T))_{T \subseteq S}$, where $\tilde{V}(T) = V(T) - u_G = \max_{a \in A} \left(\sum_{i \in T} v_i(a) - C(a) \right) - u_G$.

In the public goods provision problem, $u_G = -C(0) = 0$, and thus $\tilde{V}(T) = V(T)$ for all $T \subseteq S$. A payoff vector $u_S = (u_i)_{i \in S}$ is in *the core* if and only if $\sum_{i \in S} u_i = V(S)$ and $\sum_{i \in T} u_i \geq V(T)$ for all $T \subset S$.

4 The Main Result

This section shows our main result that the set of FRP-core allocations coincides with the set of PCPNE outcomes of the voluntary participation game. In the public goods provision problem, the above Fact (Laussel and Le Breton, 2001) implies that the second-stage CPNE outcomes coincide with the set of all core allocations of a characteristic function form game for S with $(V(T))_{T \subseteq S}$ where $V(T) = \max_{a \in A} \left(\sum_{i \in T} v_i(a) - C(a) \right)$.[12] This is nothing but Foley's (1970) core in a public goods economy for S. This observation gives us some insight into our two-stage noncooperative game. First, for each subgame played by $S' = S(\sigma^{1\prime})$, the utility outcome $u_{S'}$ must be in the core of $(V(T))_{T \subseteq S'}$. Second, given the setup of our group-formation game in the first stage, if a CPNE outcome u in a subgame S can be realized as the equilibrium outcome (on the equilibrium path), it is *necessary* that $u \in Core^{FRP}(S)$, since otherwise some member of S would deviate *in the first stage* obtaining a secured free-riding payoff. This observation is useful in our analysis of the equivalence theorem. With a careful construction of equilibrium strategies, we can show the following.

Proposition 3. If an allocation $(S, a^*(S), u)$ is in the FRP-Core, then there is a PCPNE σ whose outcome is $(S, a^*(S), u)$.

We relegate a proof of Proposition 3 to Appendix B (with some preliminary analyses in Appendix A). Here, we briefly describe how to construct PCPNE σ.

[12]Indeed, CPNE and strong Nash equilibrium (Aumann, 1959) with weakly improving deviations are equivalent in a common agency game with the no-rent property. See Konishi, Le Breton, and Weber (1999).

First, in defining σ, we need to assign a CPNE utility profile to every subgame that corresponds to a coalition $S \subseteq N$. Since the second-stage strategy profile is described by utility allocations assigned to each subgame, we partition the set of subgames (expressed in terms of active players) $\mathcal{S} = \{S \in 2^N : S \neq \emptyset\}$ into three categories: (i) $\mathcal{S}_1 = \{S^*\}$ on the equilibrium path, which is the contribution group formed in equilibrium, (ii) $\mathcal{S}_2 = \{S \in \mathcal{S} : S \cap S^* = \emptyset\}$, and (iii) $\mathcal{S}_3 = \{S \in \mathcal{S} \backslash \mathcal{S}_1 : S \cap S^* \neq \emptyset\}$. As Laussel and Le Breton (2001) show, a CPNE outcome in a subgame S' corresponds to a core allocation for S'. To support the equilibrium path $(S^*, a^*(S^*), u^*) \in Core^{FRP}$ by a PCPNE, we need to show that there is no credible deviation in the first stage. This requires careful and nontrivial assignment of a core allocation to every single subgame.

We prove Proposition 3 by contradiction. Consider a deviation from S^* by a coalition T, which leads to the formation of a new contribution group S'. As Figure 1 shows, T consists of players who change their first-stage actions ((i) and (ii) in the figure) and players who change their second-stage actions ((iv) in the figure). Suppose to the contrary that this deviation is credible. Then, for all members of T, both *profitability of deviation* and *free-riding-proofness* must be satisfied. Thus, for every player $i \in T$, the post-deviation payoff u'_i must satisfy $u'_i \geq \bar{u}_i = \max\{u^*_i, v_i(S' \backslash \{i\})\}$, where u^*_i denotes player i's payoff in the prescribed equilibrium. The case where $S' \cap S^* \neq \emptyset$ as depicted in Figure 1 is the most subtle. We show that even in such cases, if there were such a credible deviation, there would exist an allocation $(S', a^*(S'), u') \in Core^{FRP}(S')$ that Pareto-dominates $(S^*, a^*(S^*), u^*)$. By the characterization in Proposition 2, however, this contradicts the supposition that $(S^*, a^*(S^*), u^*) \in Core^{FRP}$. We show Pareto-domination by using the fact that the utility allocation assigned to subgame S' under σ is a core allocation, and construct the core allocation by the algorithm that is provided in Appendix A.

Once this direction of the relationship between the FRP-Core and PCPNE is established, the converse is trivial. The PCPNE requires free-riding-proofness, so every PCPNE must be an FRP-Core allocation for some S. Since $Core^{FRP}$ is the Pareto-frontier of $\cup_{S \subseteq N} Core^{FRP}(S)$, Proposition 3 indeed implies that any Pareto-dominated FRP-Core allocation for S can be defeated by an FRP-Core allocation, which is supported by a PCPNE.

Theorem 2. *An allocation* $(S, a^*(S), u)$ *is in the FRP-Core if and only if there is a PCPNE* σ *whose outcome is* $(S, a^*(S), u)$*.*

Proof. We prove the converse of the relationship described in Proposition 3, i.e., we show that every PCPNE σ generates an FRP-Core allocation as its outcome. It is easy to see that the outcome $(S, a^*(S), u)$ of a PCPNE σ is an FRP-Core allocation (and not just a core allocation) for S, since otherwise a player would have an incentive to free-ride in the first stage of the extensive-form game and hence the resulting allocation would not be a PCPNE. Thus, $(S, a^*(S), u) \in Core^{FRP}(S)$. Now, suppose that $u \notin Core^{FRP}$. Then, there is an FRP-Core allocation $(S', a^*(S'), u') \in Core^{FRP}$ with $u' > u$. Proposition 3 further implies that a deviation by the grand coalition N that induces $(S', a^*(S'), u')$ can attain u' with a PCPNE σ'. This means that there is a credible coalitional deviation from σ, which leads to a contradiction. Thus, every PCPNE achieves an FRP-Core allocation. □

This result depends crucially on the "comonotonicity of preferences" (Laussel and Le Breton, 2003) and perfectly nonexcludable public goods (free-riders can fully enjoy public goods). Without these assumptions, the above equivalence does not hold in general.

Although the FRP-Core is much easier to grasp than PCPNE, it may still not be clear what the FRP-Core looks like. A simple example in the next section illustrates the properties of FRP-Core allocations and thus the outcomes of PCPNE of our voluntary contribution game.

5 An Example: Linear Utility and Quadratic Cost

Let $v_i(a) = \theta_i a$ for any $i \in N$ and $C(a) = a^2/2$, where $\theta_i > 0$ is a preference parameter. In this section, we identify players by their preference parameters, i.e., $\theta_i = i$ for any $i \in N$. Then, the optimal level of the public good for group S is determined by the first-order condition $\sum_{i \in S} \theta_i - a = 0$, i.e.,

$$a^*(S) = \sum_{i \in S} \theta_i.$$

Consequently, the value of S is written as

$$\begin{aligned} V(S) &= \sum_{i \in S} \theta_i \left(\sum_{i \in S} \theta_i \right) - \frac{1}{2} \left(\sum_{i \in S} \theta_i \right)^2 \\ &= \frac{\left(\sum_{i \in S} \theta_i \right)^2}{2}. \end{aligned}$$

For an outsider $j \in N \backslash S$, the payoff is

$$v_j(a^*(S)) = \theta_j \left(\sum_{i \in S} \theta_i \right).$$

Consider the following example.

Example 1. Let $N = \{11, 5, 3, 1\}$, where $\theta_i = i$ for each $i \in N$.

First we check if the grand coalition $S = N$ is supportable. When $S = N$, we have $a^*(N) = \sum_{i \in N} i = 20$, and $V(N) = 20^2/2 = 200$. For the allocation to be free-riding-proof, each player must obtain the following payoff at the very least:

$$\begin{aligned} v_{11}(a^*(N \backslash \{11\})) &= (20 - 11) \times 11 = 99, \\ v_5(a^*(N \backslash \{5\})) &= (20 - 5) \times 5 = 75, \\ v_3(a^*(N \backslash \{3\})) &= (20 - 3) \times 3 = 51, \\ v_1(a^*(N \backslash \{1\})) &= (20 - 1) \times 1 = 19. \end{aligned}$$

The sum of all these values exceeds the value of the grand coalition $V(N)$. As a result, we can conclude $Core^{FRP}(N) = \emptyset$.

- *The FRP-Core for the grand coalition N may be empty. Thus, the FRP-Core may be suboptimal.*

Next, consider $S = \{11, 5\}$. Then, $a^*(S) = 16$, and $V(S) = 128$. To check if the FRP-Core for S is nonempty, we first check again the free-riding incentives.

$$\begin{aligned} v(a^*(S\backslash\{11\})) &= (16-11) \times 11 = 55, \\ v(a^*(S\backslash\{5\})) &= (16-5) \times 5 = 55. \end{aligned}$$

Thus, if there is an FRP-Core allocation for S, $u = (u_{11}, u_5)$ must satisfy

$$\begin{aligned} u_{11} + u_5 &= 128, \\ u_{11} &\geq 55, \\ u_5 &\geq 55, \\ u_{11} &\geq \frac{11 \times 11}{2} = 60.5, \\ u_5 &\geq \frac{5 \times 5}{2} = 12.5, \end{aligned}$$

where the last two conditions follow from the core requirement. That is, we have[13]

$$\begin{aligned} &Core(\{11, 5\}) \\ &= \{u \in \mathbb{R}^5_+ : u_{11} + u_5 = 128,\ u_{11} \geq 60.5,\ u_5 \geq 12.5, u_3 = 48,\ u_2 = 32,\ u_1 = 16\}, \end{aligned}$$

and

$$\begin{aligned} &Core^{FRP}(\{11, 5\}) \\ &= \{u \in \mathbb{R}^5_+ : u_{11} + u_5 = 128,\ u_{11} \geq 60.5,\ u_5 \geq 55, u_3 = 48,\ u_2 = 32,\ u_1 = 16\}. \end{aligned}$$

It is readily seen that $Core^{FRP}(\{11, 5\}) \neq \emptyset$, but it is a smaller set than $Core(\{11, 5\})$.

- *Free-riding-proof constraints may narrow the set of attainable core allocations for a coalition.*

Note that in this case, only the free-riding incentive constraint for player 5 is binding. It is better for player 11 to provide public goods alone than to free-ride on player 5. □

[13] For notational simplicity, we abuse notations by dropping irrelevant arguments of allocations. Thus, in this subsection, allocations are simply expressed by utility allocations.

Now, let us analyze the FRP-Core. Since the FRP-Core requires Pareto-efficiency on the union of FRP-Cores over all subsets S of the players, we first need to find the FRP-Core for each S. In general, even a minimal task of checking the nonemptiness of the FRP-Core for S is cumbersome, since the FRP-Core for S demands two almost unrelated requirements: immunity to coalitional deviation attempts and to free-riding incentives. However, it is easy to narrow down the candidates by using a necessary condition for the nonemptiness of the FRP-Core for S.

Observation. In the case of linear utility and quadratic cost, if the FRP-Core for S is nonempty, then S satisfies the following aggregate "no free-riding condition."

$$\begin{aligned} \Phi(S) &\equiv V(S) - \sum_{i \in S} \theta_i a^*(S \backslash \{i\}) \\ &= \sum_{i \in S} \theta_i a^*(S) - \frac{1}{2}(a^*(S))^2 - \sum_{i \in S} \theta_i a^*(S \backslash \{i\}) \geq 0, \end{aligned}$$

which is equivalent to

$$\sum_{i \in S} \theta_i^2 \geq \frac{1}{2} \left(\sum_{i \in S} \theta_i \right)^2.$$

The proof is straightforward and hence omitted.

By utilizing this proposition, we can characterize the FRP-Core of the public goods economy in Example 1.

Example 1. (Continued) The FRP-Core allocations are attained by groups $\{11, 5, 1\}$, $\{11, 3, 1\}$, $\{11, 5\}$, $\{11, 3\}$, and $\{5, 3\}$.

First, by applying the Observation, we find that there are 12 contribution groups that satisfy the necessary condition for the nonempty FRP-Core for S: $\{11, 5, 1\}$, $\{11, 3, 1\}$, $\{11, 5\}$, $\{11, 3\}$, $\{11, 1\}$, $\{5, 3\}$, $\{5, 1\}$, $\{3, 1\}$, $\{11\}$, $\{5\}$, $\{3\}$, and $\{1\}$.

The FRP-Core for $S = \{11, 5, 3\}$ is empty, for example. For $S = \{11, 5, 3\}$, we have $a^*(S) = 19$ and $V(S) = 180.5$. Since $11v(a^*(S \backslash \{11\})) = 88$, $5v(a^*(S \backslash \{5\}) = 70$, $3v(a^*(S \backslash \{3\})) = 48$, and $88 + 70 + 48 > 180.5$, the necessary condition for $S = \{11, 5, 3\}$ to give an FRP-Core allocation is violated. As we will see, however, $Core^{FRP}(\{11, 5, 1\})$ is not empty. Thus

$\{11, 5, 1\}$ is the group that achieves the highest level of public goods while having a nonempty FRP-Core. This analysis provides an interesting observation.

- *(Even the largest) group that achieves an FRP-Core allocation may not be consecutive.*[14]

The intuition behind this result is simple. Suppose $\Phi(S)$ is positive (say, $S = \{11, 5\}$). Now, we try to find $S' \supset S$ that still satisfies $\Phi(S') \geq 0$. If the value of $\Phi(S)$ is positive and yet not too large, then adding a player with a high θ (say, player 3) may make $\Phi(S') < 0$, since adding such a player may greatly increase $a^*(S')$, making the free-riding problem more severe. By contrast, adding a player with a low θ (say, player 1) does not make the free-rider problem too severe, so $\Phi(S') \geq 0$ may be satisfied relatively easily.

Among the above 12 groups, it is easy to see that groups $\{5, 1\}$, $\{3, 1\}$, $\{11\}$, $\{5\}$, $\{3\}$, and $\{1\}$ do not survive the test of Pareto-domination. For example, consider $S' = \{11, 5\}$ and $u' = (73, 55, 48, 16) \in Core^{FRP}(\{11, 5\})$. This is the best allocation for player 11 in $Core^{FRP}(\{11, 5\})$ as the characterization of $Core^{FRP}(\{11, 5\})$ in the above indicates. Players other than 11 and 5 are free-riders, and their payoffs are directly generated from $a^*(\{11, 5\}) = 16$. Now it is straightforward to see that the allocation u' dominates all allocations for the above six groups; public goods provision levels of those groups are insufficient compared with $a^*(\{11, 5\}) = 16$.

By contrast, $\{5, 3\}$ is not dominated by any FRP-Core allocations for any contribution group. We can show that player 11 can obtain at most 73 in a FRP-Core allocation for any $S \ni 11$, whereas she obtains 88 by free-riding on $\{5, 3\}$. Thus, player 11 would not join a deviation. Without player 11's cooperation, no free-riding core allocation that dominates those of $\{5, 3\}$ can be realized.

Similarly, FRP-Core allocations for $S = \{11, 1\}$ are dominated by the one for $S' = \{11, 5\}$. Under $S = \{11, 1\}$, player 5 obtains 60, but S' can attain $u' = (63, 65, 48, 16)$. Free-riding-proof core allocations for $\{11, 3, 1\}$ and $\{11, 3\}$ cannot be beaten, however, by the ones for

[14] Although the context and approach are very different from ours, the formation of such nonconsecutive coalitions has attracted tremendous interests in the fields of political science and sociology. For a game-theoretical treatment of this line of literature (known and "Gamson's law"), see Le Breton, et al. (2008).

$S' = \{11,5\}$; player 5, for example, gets 70 even under $\{11,3\}$ while she would obtain at most 67.5 under $S' = \{11,5\}$, as we can see from $Core^{FRP}(\{11,5\})$ derived above.

Finally, consider $S = \{11,5\}$, $\{11,3\}$. The FRP-Core allocations for $S = \{11,5\}$ are characterized by $u_{11} + u_5 = 128$, $u_{11} \geq 60.5$ and $u_5 \geq 55$, with $u_3 = 48$ and $u_1 = 16$. Now, consider $S' = \{11,5,1\}$, for which the FRP-Core allocations are characterized by $u'_{11} + u'_5 + u'_1 = 144.5$, $u'_{11} + u'_5 \geq 128$, $u'_{11} \geq 66$, $u'_5 \geq 60$, and $u'_1 \geq 16$, with $u'_3 = 51$. ($u'_5 + u'_1 \geq 18$ and $u'_{11} + u'_1 \geq 72$ are satisfied because $u'_{11} \geq 66$, $u'_5 \geq 60$, and $u'_1 \geq 16$.) Here, S' can attain $u'_{11} + u'_5 = 144.5 - 16 = 128.5$ as long as $u'_{11} \geq 66$ and $u'_5 \geq 60$. Thus, if $u \in Core^{FRP}(\{11,5\})$ satisfies $u_{11} + u_5 = 128$, $60.5 \leq u_{11} \leq 68.5$, and $55 \leq u_5 \leq 62.5$, then u is improved upon by an allocation in $Core^{FRP}(\{11,5,1\})$. However, if $u \in Core^{FRP}(\{11,5\})$ satisfies $u_{11} + u_5 = 128$, $u_{11} > 68.5$, or $u_5 > 62.5$, then u cannot be improved upon by group $\{11,5,1\}$. The FRP-Core allocations for $S = \{11,3\}$ have a similar property with possible deviations by group $S' = \{11,3,1\}$. This phenomenon illustrates another interesting observation:

- *An expansion of a group definitely increases the total value of the group, while it gives less flexibility in allocating the benefits among the group members since free-riding incentives increase as the level of the public goods provision rises. As a result, some unequal FRP-Core allocations for the original group may not be improved upon by the group expansion.*

In summary, the FRP-Core is the *union* of the following sets of allocations attained by the five different groups.

1. $S = \{11,5,1\}$; $a^*(S) = 17$ and all FRP-Core allocations for S are included:

$$\begin{aligned} & Core^{FRP}(\{11,5,1\}) \\ = \; & \{u \in \mathbb{R}^5_+ : u_{11} + u_5 + u_1 = 144.5,\ u_3 = 51,\ u_{11} \geq 66,\ u_5 \geq 60,\ u_1 \geq 16.\} \end{aligned}$$

2. $S = \{11,3,1\}$; $a^*(S) = 15$ and all FRP-Core allocations for S are included:

$$\begin{aligned} & Core^{FRP}(\{11,3,1\}) \\ = \; & \{u \in \mathbb{R}^5_+ : u_{11} + u_3 + u_1 = 112.5,\ u_5 = 75, u_{11} \geq 60.5,\ u_3 \geq 36,\ u_1 \geq 14\}. \end{aligned}$$

3. $S = \{11, 5\}$; $a^*(S) = 16$ and only a subset of FRP-Core allocations for S is included:

$$\begin{aligned}&\left\{u \in Core^{FRP}(\{11,5\}) : u_{11} > 68.5 \text{ or } u_5 > 62.5\right\}\\ = \;&\{u \in \mathbb{R}^5_+ : u_{11} + u_5 = 128,\; u_3 = 48,\; u_1 = 16, 68.5 < u_{11} \leq 73 \text{ or } \; 62.5 < u_5 \leq 67.5\}\end{aligned}$$

4. $S = \{11, 3\}$; $a^*(S) = 14$ and only a subset of FRP-Core allocations for S is included:

$$\begin{aligned}&\left\{u \in Core^{FRP}(\{11,3\}) : u_{11} > 62.5\right\}\\ = \;&\{u \in \mathbb{R}^5_+ : u_{11} + u_3 = 98,\; u_5 = 70,\; u_1 = 14, 62.5 < u_{11} \leq 65\}\end{aligned}$$

5. $S = \{5, 3\}$; $a^*(S) = 8$ and all FRP-Core allocations for S are included:

$$\begin{aligned}&Core^{FRP}(\{5,3\})\\ = \;&\{u \in \mathbb{R}^5_+ : u_5 + u_3 = 32,\; u_{11} = 88,\; u_1 = 8, u_5 \geq 15,\; u_3 \geq 15\}\end{aligned}$$

□

Before closing this section, we compare the FRP-Core allocations with a Nash equilibrium of a simultaneous-move voluntary public goods provision game studied by Bergstrom, Blume, and Varian (1986). Each player i chooses her monetary contribution $m_i \geq 0$ to finance a public good. The public goods provision level is given by $a(m) = \sqrt{2\sum_{i\in N} m_i}$ reflecting the cost function of public goods production $C(a) = a^2/2$. Consider player i. Given that others contribute M_{-i} in total, player i chooses m_i so as to maximize $\theta_i\sqrt{2\,(m_i + M_{-i})} - m_i$. The best response for player i is $m_i^* = \max\{(\theta_i^2/2) - M_{-i}, 0\}$. It is easy to see that in our example, only player 11 contributes, so the public goods provision level is 11.[15] Thus, by forming a contribution group in the first stage, it is possible to increase the equilibrium level of the public goods provision. But it is also possible that the level of public goods provision is lower than the Nash equilibrium provision level of the standard voluntary contribution game, as we have found that group $\{5, 3\}$ achieves some FRP-Core allocations in our example.

[15] Contribution is made only by the highest willingness-to-pay player. This observation is true for all quasi-linear utility players (with no income effect).

- *There may be FRP-Core allocations that achieve lower public goods provision levels than the Nash equilibrium outcome of a simple voluntary contribution game studied by Bergstrom, Blume, and Varian (1986).*

This occurs because in our setup, player 11 can commit to being an outsider in the first stage, which cannot happen in a simultaneous-move voluntary contribution game. Finally, needless to say, we have:

- *The FRP-Core may be a highly nonconvex set, as different allocations may be realized by different contribution groups.*

6 Replicated Economies

In this section, we analyze whether or not public goods provision and the participation rate decrease as the economy is replicated.

There is a tricky issue in replicating a (pure) public goods economy. If the set of consumers is simply replicated, the amount of resources in the economy grows to infinity, while maintaining the same cost function for public goods production. Following Milleron's (1972) method, Healy (2007) makes each consumer's endowment shrink proportionally to the population as the economy is replicated to overcome this problem; consumers' preferences are also modified in the replication process.[16] We adopt the same preference modification in the replication of a quasi-linear economy. We shrink each consumer's willingness-to-pay proportionally as the economy is replicated. This method of replication is natural for a quasi-linear economy, since the aggregate willingness-to-pay and cost functions stay the same in the replication process.

The original economy is a list $E = (N, (v_i)_{i \in N}, C)$. Let $r = 1, 2, 3, ...$ be a natural number. The rth replica of E is a list $E^r = (N^r, (v^r_{i_q})_{i \in N, q=1,...,r}, C)$, where $N^r = \cup_{i \in N} \{i_1, ..., i_r\}$ and $v^r_{i_q}(a) = v^r_i(a) = \frac{1}{r} v_i(a)$ for all $q = 1, .., r$.[17] Let a characteristic function form game generated from E^r be V^r.

[16]Conley (1994) uses a different definition of replicated economy and investigates the convergence of the core.

[17]Let x and a denote the consumption level of a private good and the level of a public good, and let $\succeq_i$

We analyze FRP-Core allocations $(S, a^*(S), u^*)$ of the characteristic function form game V^r by focusing on the free-riding-proofness condition. Note that for any r, and for any $S \subseteq N^r$, the public goods provision level $a = a^*(S)$ is determined so that the sum of willingness-to-pay across all members of S equals the marginal cost of public goods provision, i.e., $\sum_{i_q \in S} v_{i_q}^{r\prime}(a) = C'(a)$. Furthermore, we need $\sum_{i_q \in S} \left(v_{i_q}^r(a^*(S)) - v_{i_q}^r(a^*(S \backslash \{i_q\}) \right) \geq C(a^*(S))$ in order to satisfy the free-riding-proofness, where the terms in the parentheses on the left-hand side indicate how much each player can pay without sacrificing free-riding-proofness. Let $m_i(S) \in \{0, \cdots, r\}$ denote the number of type i players in S. Then, the above necessary condition for free-riding-proofness can be rewritten as

$$\sum_{i \in N} m_i(S) \left(v_i^r(a^*(S)) - v_i^r(a^*(S \backslash \{i_q\})) \right) \geq C(a^*(S)),$$

where it should be understood that $S \setminus \{i_q\}$ denotes the set of all players but *one* player of type i in S. Or equivalently,

$$\sum_{i \in N} \frac{m_i(S)}{r} \left[v_i(a^*(S)) - v_i(a^*(S \backslash \{i_q\})) \right] \geq C(a^*(S)). \tag{1}$$

Now, consider the kth replication, where $k = 1, 2, \cdots$, of this rth replica of the original economy, which implies that each player in the rth replica of the original economy is divided into k players. Let S^k be a coalition in this $k \times r$th replica economy that contains all k replica players of all members of S in rth replica economy. Obviously, $a^*(S)$ in rth replica economy equals $a^*(S^k)$ in $k \times r$th replica economy. However, although the coefficients satisfy $m_i(S)/r = m_i(S^k)/(kr)$, $a^*(S^k \backslash \{i\})$ converges to $a^*(S^k) = a^*(S)$ as k goes to infinity. Thus, the $k \times r$th replica economy's counterpart of inequality (1) would be violated at some point. Formally, we have the following result.

Proposition 4. Suppose that C and v_i are twice continuously differentiable for any $i \in N$ with (i) $C(0) = 0$, $C'(a) > 0$, $C''(a) > 0$, and $\lim_{a \to 0} C'(a) = 0$, and (ii) $v_i'(a) > 0$ and $v_i''(a) \leq 0$ for all $i \in N$. Then, for any $\bar{a} > 0$, there exists a natural number $\bar{r}(\bar{a})$ such that for any $r \geq \bar{r}(a)$, $a^*(S^*) < \bar{a}$ holds for any $(S^*, a^*(S^*), u^*) \in Core^{FRP}(V^r)$.

and $\succeq_i^r$ be preference relations in the original and rth replica economy, respectively. According to Milleron's (1972) preference modification, relation $\succeq_i^r$ is generated such that $(x, a) \succeq_i^r (x', a')$ if $(rx, a) \succeq_i (rx', a')$. In the quasi-linear economy where $\succeq_i$ is described by the utility function $x + v_i(a)$, this implies $v_i^r(a) = v_i(a)/r$.

Together with Theorem 1, Proposition 4 immediately implies the following theorem.

Theorem 3. *Suppose that C and v_i are twice continuously differentiable for any $i \in N$ with (i) $C(0) = 0$, $C'(a) > 0$, $C''(a) > 0$, and $\lim_{a \to 0} C'(a) = 0$, and (ii) $v_i'(a) > 0$ and $v_i''(a) \leq 0$ for all $i \in N$. Then, the public goods provision levels for all FRP-Core allocations shrink to zero as the economy is replicated.*

Although this result has some similarity to the main result of Healy (2007), the models and the objectives are very different. Unlike our model, Healy requires that all players (voluntarily) participate in equilibrium, while he does not ask contribution groups to achieve efficient provision of public goods. Thus, the reasons for the convergence are very different in his and our papers. Note also that unlike Theorems 1 and 2, Theorem 3 (and Proposition 4) relies on the concavity and convexity of utility and cost functions, respectively, as well as the differentiability of them.

7 Conclusion

This paper has added players' participation decisions to a (pure) public goods provision problem. We propose a free-riding-proof core (FRP-Core), which is a hybrid solution concept based on credibility of coalitional deviations. The FRP-Core is always nonempty in a public goods economy but does not usually achieve global efficiency. The FRP-Core has support from both cooperative and noncooperative games. In particular, it is equivalent to the set of perfectly coalition-proof Nash equilibria (Bernheim, Peleg, and Whinston, 1987) of a dynamic game with participation decisions followed by a common agency contribution game. With a simple example, we have found that the equilibrium contribution group may not be consecutive (with respect to players' willingness-to-pay), and the public good may be under-provided (compared with the case of voluntary contribution game studied by Bergstrom, Blume, and Varian, 1986, for example). Furthermore, the public goods provision level decreases to zero as the economy grows.

Although we have restricted our analysis to the public goods problem with transferable utility (assuming that all players have quasi-linear utilities), we can extend our analysis to a Gorman-form utility function (Bergstrom and Cornes, 1983) to allow the (positive) income effect for the public goods.[18] Suppose that player i's preferences are represented by a utility function of the form $u_i(a, x) = \alpha(a)x + \beta_i(a)$ for all $i = 1, ..., n$, while the government's utility is $u_0(a, x) = \alpha(a)\,(x - C(a))$. As long as players have enough endowments such that their budget constraints would not be binding, the characterization results of Bernheim and Whinston (1986) extend to this utility specification. We can show that if $\alpha(a)$ is nondecreasing, and the ratio $\beta_i(a)/\alpha(a)$ is nondecreasing for all $i \in N$, the efficient public good provision level $a^*(S)$ is nondecreasing with respect to group expansion, and hence the resulting game $(V(S))_{S\subseteq N}$ is a convex game. Thus, our equivalence theorem continues to hold in this class of public goods economies. Relaxing the assumption on utility functions (even further) is of interest, since then we would be able to examine how the income distribution across players affects free-riding incentives and equilibrium public goods provision levels, for example. Indeed, Theorem 1 of our analysis extends to a general NTU game. To obtain our main equivalence result, however, we have appealed to the results obtained by Bernheim and Whinston (1986), who analyze TU games. Thus, we first need to extend their analysis to NTU games to examine the equivalence between the FRP-Core and PCPNE. We leave this interesting and nontrivial exercise to our future research.

Appendix A: Preliminary Analysis on the Core of Convex Games

In this appendix, we list a few useful preliminary results on the core of convex games. In our public goods (comonotonic) domain, the characteristic-function game generated from a (public goods) economy is convex. Let $V : 2^N \to \mathbb{R}$ with $V(\emptyset) = 0$ be a characteristic-function form game. Game V is *convex* if $V(S \cup T) + V(S \cap T) \geq V(S) + V(T)$ for all pairs of subsets S and T of N. The *core* of game V is $Core(N, V) = \{u \in \mathbb{R}^N : \sum_{i\in N} u_i = V(N)$ and $\sum_{i\in S} u_i \geq V(S)$ for all $S \subset N\}$. Shapley (1971) analyzes the properties of the core of

[18] We thank a referee for bringing our attention to this possible extension.

convex games in detail. One of his results that is useful for us is the following.

Property 1. (Shapley, 1971) Let $\omega : \{1, \cdots, |N|\} \rightarrow N$ be an arbitrary bijection, and let $u_{\omega(1)} = V(\{\omega(1)\})$, $u_{\omega(2)} = V(\{\omega(1), \omega(2)\}) - V(\{\omega(1)\})$,..., and $u_{\omega(|N|)} = V(N) - V(N \backslash \{\omega(|N|)\})$. Then, $u = (u_i)_{i \in N} \in Core(N, V)$, and the set of all such allocations forms the set of vertices of $Core(N, V)$.

Now, we consider a reduced game, in which outsiders always join coalitions and walk away with the payoffs they could obtain by forming their own coalition. Let T be a proper subset of N. A reduced game of V on T is $\tilde{V}_T : 2^T \rightarrow \mathbb{R}$ such that $\tilde{V}_T(S) = V(S \cup (N \backslash T)) - V(N \backslash T)$ for all $S \subseteq T$. We have the following result.

Property 2. Suppose that $V : N \rightarrow \mathbb{R}$ is a convex game. Let $u_{N \backslash T} = (u_i)_{i \in N \backslash T}$ be a core allocation of a game $V : N \backslash T \rightarrow \mathbb{R}$. Then, $u_T \in Core(T, \tilde{V}_T)$ if and only if $(u_T, u_{N \backslash T}) \in Core(N, V)$.

Proof. First, we show that $u_T \in Core(T, \tilde{V}_T)$ if $(u_T, u_{N \backslash T}) \in Core(N, V)$. Since $(u_T, u_{N \backslash T}) \in Core(N, V)$, $\sum_{i \in S \cup (N \backslash T)} u_i \geq V(S \cup (N \backslash T))$ holds for all $S \subset T$. Rewriting this, we have $\sum_{i \in S} u_i \geq V(S \cup (N \backslash T)) - \sum_{i \in N \backslash T} u_i = V(S \cup (N \backslash T)) - V(N \backslash T) = \tilde{V}_T(S)$. Thus, $u_T \in Core(T, \tilde{V})$.

Second, we show that $u_T \in Core(T, \tilde{V}_T)$ implies $(u_T, u_{N \backslash T}) \in Core(N, V)$. Suppose this is not the case. Then, there is $S \subset N$ such that

$$V(S) > \sum_{i \in S} u_i = \sum_{i \in S \cap T} u_i + \sum_{i \in S \cap (N \backslash T)} u_i. \tag{2}$$

Since $u_T \in Core(T, \tilde{V})$ and V is a convex game, we have $\sum_{i \in S \cap T} u_i \geq V(S \cup (N \backslash T)) - V(N \backslash T) \geq V(S) - V(S \cap (N \backslash T))$. Substituting this inequality into (2), we have $V(S) > V(S) - V(S \cap (N \backslash T)) + \sum_{i \in S \cap (N \backslash T)} u_i$, which leads to a contradiction since $u_{N \backslash T} \in Core(N \backslash T, V)$ implies $\sum_{i \in S \cap (N \backslash T)} u_i \geq V(S \cap (N \backslash T))$. $\square$

Now, we rewrite the core. Let $u = (u_i)_{i \in N}$ be an arbitrary utility vector. Let

$$\begin{aligned} \mathcal{Q}^+(u) &= \{S \in 2^N : \sum_{j \in S} u_j > V(S)\}, \\ \mathcal{Q}^0(u) &= \{S \in 2^N : \sum_{j \in S} u_j = V(S)\}, \\ \mathcal{Q}^-(u) &= \{S \in 2^N : \sum_{j \in S} u_j < V(S)\}. \end{aligned}$$

That is, sets $\mathcal{Q}^+(u)$ and $\mathcal{Q}^-(u)$ denote the collections of coalitions in which players as a whole are satisfied and unsatisfied (in the strict sense) with the utility vector u, respectively. The set $\mathcal{Q}^0(u)$ is the collection of coalitions in which players are just indifferent collectively between deviating and not deviating. Obviously, a utility vector u is in the core, i.e., $u \in Core(N, V)$, if and only if $\mathcal{Q}^-(u) = \emptyset$ (or $S \in \mathcal{Q}^+(u) \cup \mathcal{Q}^0(u)$ for all $S \in 2^N$) and $N \in \mathcal{Q}^0(u)$. Let $\eta(S, u) \equiv [V(S) - \sum_{i \in S} u_i]/|S|$ be the *(per capita) shortage of payoff* for coalition S for any $S \in \mathcal{Q}^-(u)$. Let

$$\mathcal{Q}^-_{\max}(u) \equiv \{S \in \mathcal{Q}^-(u) : \eta(S, u) \geq \eta(S', u) \text{ for all } S' \in \mathcal{Q}^-(u)\},$$

and

$$Q^-_{\max}(u) = \cup_{S \in \mathcal{Q}^-_{\max}(u)} S.$$

Using the above definitions, we now construct an algorithm that starts from an arbitrary utility vector u and terminates with a core allocation $\hat{u}$.

Algorithm. Let $u \in \mathbb{R}^N$ and let $V : N \to \mathbb{R}$ be a convex game. Let $u(t)$ be the utility vector at stage $t \in \mathbb{R}_+$, and $u(0) = u$ (the initial value).

(a) Suppose $\mathcal{Q}^-(u) = \emptyset$. Then, $2^N \backslash \{\emptyset\} = \mathcal{Q}^0(u) \cup \mathcal{Q}^+(u)$. If $N \in \mathcal{Q}^0(u(0))$, then the algorithm terminates immediately. Otherwise, $\sum_{i \in N} u_i > V(N)$ holds, and we reduce each u_i for $i \in N \backslash (\cup_{S \in \mathcal{Q}^0(u)} S)$ continuously at a common speed as t increases.[19] Since

[19] It follows from the definition of a convex game that $\cup_{Q \in \mathcal{Q}^0(u)} Q = N$ implies $N \in \mathcal{Q}^0(u)$. To prove this claim, it suffices to show that if $T, T' \in \mathcal{Q}^0(u)$, then $T \cup T' \in \mathcal{Q}^0(u)$ when $\mathcal{Q}^-(u) = \emptyset$ as is assumed. We have from the definition of a convex game that $V(T \cup T') + V(T \cap T') \geq V(T) + V(T') = \sum_{i \in T \cup T'} u_i + \sum_{i \in T \cap T'} u_i$. Since $T \cap T' \in \mathcal{Q}^0(u) \cup \mathcal{Q}^+(u)$, $\sum_{i \in T \cap T'} u_i \geq V(T \cap T')$. Together with the above inequality, this implies $V(T \cup T') \geq \sum_{i \in T \cup T'} u_i$. Since $\mathcal{Q}^-(u) = \emptyset$, $T \cup T' \in \mathcal{Q}^0(u)$.

all elements in $\mathcal{Q}^0(u)$ continue to be in $\mathcal{Q}^0(u(t))$, while some of elements of $\mathcal{Q}^+(u(t))$ switch to $\mathcal{Q}^0(u(t))$ in the process, $\mathcal{Q}^0(u(t))$ monotonically expands as t increases. Thus, $N \in \mathcal{Q}^0(u(\hat{t}))$ occurs at some stage $\hat{t}$. Then we terminate the process. The final outcome is $\hat{u} = u(\hat{t})$.

(b) Suppose $\mathcal{Q}^-(u) \neq \emptyset$. There are two phases, starting with phase 1.

i. Phase 1: Start with $u(0) = u$. For all $i \in Q^-_{\max}(u(t))$, increase u_i continuously at a common speed. Terminate this phase of the algorithm when $Q^-_{\max}(u(t)) = \emptyset$ (or $\mathcal{Q}^-(u(t)) = \emptyset$), and call such t as $\tilde{t}$.[20]

ii. Phase 2: Now, $\mathcal{Q}^-(u(\tilde{t})) = \emptyset$. Then, we go to the procedure in (a), and we reach a final outcome $\hat{u} = u(\hat{t})$ when $N \in \mathcal{Q}^0(u(\hat{t}))$ occurs. □

Let $Q^0(u) \equiv \cup_{S \in \mathcal{Q}^0(u)} S$, and define

$$\begin{aligned} W &\equiv \{i \in N : \exists t \geq 0 \text{ with } i \in Q^-_{\max}(u(t)) \text{ in phase 1 of case (b)}\}, \\ I &\equiv \{i \in N : i \in Q^0(u(0)) \text{ in case (a), or } i \in Q^0(u(\tilde{t}))\backslash W \text{ in case (b)}\}, \\ L &\equiv \{i \in N : i \notin Q^0(u(0)) \text{ in case (a), or } i \notin Q^0(u(\tilde{t})) \text{ in case (b)}\}. \end{aligned}$$

These sets will be shown to be collections of players who gain, remain indifferent, and lose in the above algorithm relative to the initial value u, respectively. By the construction of the algorithm, the following Lemma is straightforward.

Lemma 1. Set N is partitioned into W, I, and L: $\hat{u}_i > u_i$ for all $i \in W$, $\hat{u}_i = u_i$ for all $i \in I$, and $\hat{u}_i < u_i$ for all $i \in L$.

Proof. Note that the payoff for any player in W does not change in phase 2 of case (b) as $W \subseteq \cup_{S \in \mathcal{Q}^0(u(\tilde{t}))} S$. Thus, for all $i \in W$, $\hat{u}_i > u_i$. Given this, the rest is obvious. □

This lemma identifies the winners, unaffected players, and losers of the algorithm as sets W, I, and L, respectively.

[20] This process guarantees that every player $i \in Q^-_{\max}(u(t))$ at some stage $t \in [0, \tilde{t}]$ must belong to some $S' \in \mathcal{Q}^0(u(\tilde{t}))$ at the end of phase 1.

Lemma 2. Consider the above algorithm. In phase 1 of case (b), $Q^-_{\max}(u(t))$ monotonically expands as t increases for $t \in [0, \tilde{t})$. This phase terminates with $\mathcal{Q}^-(u(\tilde{t})) = \emptyset$. Moreover, $W = \lim_{t \to \tilde{t}} Q^-_{\max}(u(t)) \in \mathcal{Q}^0(u(\tilde{t}))$, and $W \in \mathcal{Q}^0(u(\hat{t}))$.

Proof. As t increases, the payoffs of all members of $Q^-_{\max}(u(t))$ increase at the same speed; thus for any $S \in \mathcal{Q}^-_{\max}(u(t))$, $\eta(S, u(t))$ decreases at the same speed. Note that for all other coalitions $T \notin \mathcal{Q}^-_{\max}(u(t))$, $\eta(T, u(t))$ decreases at a slower pace (if $T \cap Q^-_{\max}(u(t)) \neq \emptyset$) or stays constant (if $T \cap Q^-_{\max}(u(t)) = \emptyset$). Therefore, $Q^-_{\max}(u(t))$ monotonically expands as t increases. This monotonic utility-raising process continues until $\mathcal{Q}^-(u(t)) = \emptyset$ realizes at $t = \tilde{t}$. Since $Q^-_{\max}(u(t))$ monotonically expands, $W = \lim_{t \to \tilde{t}} Q^-_{\max}(u(t))$ holds.

Now, we will show $Q^-_{\max}(u) = \cup_{S \in \mathcal{Q}^-_{\max}(u)} S \in \mathcal{Q}^-_{\max}(u)$, which proves $W \in \mathcal{Q}^0(u(\tilde{t}))$ and $W \in \mathcal{Q}^0(u(\hat{t}))$ (in phase 2 of case (b), payoffs of players in W are not affected). Let $S_1, S_2 \in \mathcal{Q}^-_{\max}(u)$ with $S_1 \neq S_2$. Let

$$\bar{\eta} \equiv \frac{V(S_1) - \sum_{i \in S_1} u_i}{|S_1|} = \frac{V(S_2) - \sum_{i \in S_2} u_i}{|S_2|}.$$

By convexity, it follows that

$$\begin{aligned} V(S_1 \cup S_2) + V(S_1 \cap S_2) &\geq V(S_1) + V(S_2) \\ &= \bar{\eta}\,(|S_1| + |S_2|) + \sum_{i \in S_1} u_i + \sum_{i \in S_2} u_i. \end{aligned}$$

Since

$$\frac{V(S_1 \cap S_2) - \sum_{i \in S_1 \cap S_2} u_i}{|S_1 \cap S_2|} \leq \bar{\eta},$$

we have

$$\begin{aligned} V(S_1 \cup S_2) &\geq \bar{\eta}\,(|S_1| + |S_2|) + \sum_{i \in S_1} u_i + \sum_{i \in S_2} u_i - V(S_1 \cap S_2) \\ &\geq \bar{\eta}\,(|S_1| + |S_2| - |S_1 \cap S_2|) + \sum_{i \in S_1} u_i + \sum_{i \in S_2} u_i - \sum_{i \in S_1 \cap S_2} u_i, \end{aligned}$$

or

$$\frac{V(S_1 \cup S_2) - \sum_{i \in S_1 \cup S_2} u_i}{|S_1 \cup S_2|} \geq \bar{\eta}.$$

Thus, $S_1 \cup S_2 \in \mathcal{Q}^-_{\max}(u)$. Repeated application of the same argument proves $Q^-_{\max}(u) \in \mathcal{Q}^-_{\max}(u)$. □

Lemma 3. Starting from any initial value $u \in \mathbb{R}^N$, this algorithm terminates with a core allocation $\hat{u} \in Core(N, V)$.

Proof. First, we show that case (a) terminates with a core allocation. To this end, we need only show that $\cup_{S \in \mathcal{Q}^0(u)} S \neq N$ whenever $\sum_{i \in N} u_i > V(N)$ (otherwise, the algorithm terminates with an infeasible u). Suppose to the contrary that $\sum_{i \in N} u_i > V(N)$, while $\cup_{S \in \mathcal{Q}^0(u)} S = N$ in case (a). Let $S_1, S_2, ..., S_K \in \mathcal{Q}^0(u)$ be distinct subsets of N with $\cup_{k=1}^K S_k = N$. Then, we have $\sum_{i \in S_1} u_i = V(S_1)$ and $\sum_{i \in S_2} u_i = V(S_2)$. By convexity, $V(S_1 \cup S_2) + V(S_1 \cap S_2) \geq V(S_1) + V(S_2) = \sum_{i \in S_1} u_i + \sum_{i \in S_2} u_i$ holds. By the construction of the algorithm, $S_1 \cap S_2 \in \mathcal{Q}^0(u)$ or $S_1 \cap S_2 \in \mathcal{Q}^+(u)$, i.e., $V(S_1 \cap S_2) \leq \sum_{i \in S_1 \cap S_2} u_i$ holds. Thus, we have $V(S_1 \cup S_2) \geq \sum_{i \in S_1 \cup S_2} u_i$. Applying the same argument to $S_1 \cup S_2$ and S_3, we have $V(S_1 \cup S_2 \cup S_3) \geq \sum_{i \in S_1 \cup S_2 \cup S_3} u_i$, since $(S_1 \cup S_2) \cap S_3 \subset S_3$ implies $(S_1 \cup S_2) \cap S_3 \in \mathcal{Q}^0(u)$ or $(S_1 \cup S_2) \cap S_3 \in \mathcal{Q}^+(u)$. Repeated application of the same argument generates $V(N) = V(\cup_{k=1}^K S_k) \geq \sum_{i \in \cup_{k=1}^K S_k} u_i = \sum_{i \in N} u_i$. This is a contradiction. Thus, in case (a), the algorithm terminates with a feasible allocation. Since $u(t)$ changes continuously, $N \in \mathcal{Q}^0(\hat{u})$ holds, and $\hat{u} \in Core(N, V)$.

Now, it follows from Lemma 2 that phase 1 of case (b) terminates with $\mathcal{Q}^-(\tilde{u}) = \emptyset$. Thus, the same argument as in case (a) applies to phase 2 of case (b), leading to the conclusion that $\hat{u} \in Core(N, V)$ also in case (b). $\square$

Appendix B: Proofs

Proof of Proposition 1.

First, we show that $(T, a^*(T), u') \in Core(T)$ blocking $(S, a^*(S), u) \in Core(S)$ implies $a^*(S) < a^*(T)$. Since $(S, a^*(S), u) \in Core(S)$, $S \supseteq T$ cannot happen. Thus, we have either (i) $S \subsetneq T$, or (ii) $S \nsubseteq T$ and $S \nsupseteq T$. Case (i) implies $a^*(S) < a^*(T)$, since if $a^*(S) = a^*(T)$, blocking cannot occur (core allocations are efficient for the contribution group). Thus, consider case (ii). Note that for all $i \in T \backslash S$, we have $u'_i \geq u_i = v_i(a^*(S))$.

Suppose to the contrary that $a^*(S) \geq a^*(T)$ holds. Since

$$\sum_{j \in T} u'_j = \sum_{j \in T} v_j(a^*(T)) - C(a^*(T)),$$

we have

$$\begin{aligned}
\sum_{j \in T \cap S} u'_j &= \sum_{j \in T} v_j(a^*(T)) - C(a^*(T)) - \sum_{i \in T \setminus S} u'_i \\
&= \sum_{j \in T \cap S} v_j(a^*(T)) - C(a^*(T)) - \left(\sum_{i \in T \setminus S} u'_i - \sum_{i \in T \setminus S} v_i(a^*(T)) \right) \\
&\leq \sum_{j \in T \cap S} v_j(a^*(T \cap S)) - C(a^*(T \cap S)) - \left(\sum_{i \in T \setminus S} u'_i - \sum_{i \in T \setminus S} v_i(a^*(T)) \right) \\
&= V(T \cap S) - \left(\sum_{i \in T \setminus S} u'_i - \sum_{i \in T \setminus S} v_i(a^*(T)) \right).
\end{aligned}$$

Since $(T, a^*(T), u') \in Core(T)$, the content of the above parenthesis is nonpositive.

$$\sum_{i \in T \setminus S} u'_i \leq \sum_{i \in T \setminus S} v_i(a^*(T)).$$

Since $a^*(S) \geq a^*(T)$, we have $u'_i \geq v_i(a^*(S)) \geq v_i(a^*(T))$ for all $i \in T \setminus S$. Thus, we conclude

$$u'_i = v_i(a^*(S)) = v_i(a^*(T)) \text{ for all } i \in T \setminus S, \text{ and } a^*(S) = a^*(T).$$

This also implies

$$\sum_{j \in T \cap S} u'_j = V(T \cap S).$$

Since $u_i = v_i(a^*(S)) = u'_i$ for all $i \in T \setminus S$, none in coalition T is strictly better off by this deviation $(T, a^*(T), u')$. This is a contradiction. Thus, $a^*(T) > a^*(S)$ must hold.

Now, with the above result, it is easy to prove the rest. Since $(T, a^*(T), u') \in Core(T)$ blocks $(S, a^*(S), u) \in Core(S)$, and $a^*(T) > a^*(S)$, it is clear that $(T, a^*(T), u')$ weakly Pareto-dominates $(S, a^*(S), u)$. □

Proof of Proposition 2.

Pick a coalition $\bar{S}$ that achieves the highest level of public goods provision among the coalitions that support $Core^{FRP}$. There exists such $\bar{S}$, since the number of coalitions is finite. By Proposition 1, if $(S, a^*(S), u) \in Core^{FRP}(S)$ is weakly blocked by $(T, a^*(T), u') \in$

$Core^{FRP}(T)$, then $a^*(S) < a^*(T)$. Thus, no allocation in $Core^{FRP}(\bar{S})$ is not weakly blocked by any other allocations in $\cup_{S' \in 2^N} Core^{FRP}(S')$. Thus, $Core^{FRP}(\bar{S}) \subseteq \overline{\cup_{S' \in 2^N} Core^{FRP}(S')} = Core^{FRP}$. This implies that $Core^{FRP}$ is nonempty. $\square$

Proof of Theorem 1.

Let $Core \left\langle X^{FRP}, S \right\rangle = \{(S, a^*(S), u) \in X^{FRP} : \forall T \subseteq S, \forall (T, a^*(T), u') \in X^{FRP}, \exists i \in T \text{ s.t. } u'_i < u_i\}$. This is a collection of FRP and efficient allocations for S that are immune to *nested* FRP and efficient deviations. We first claim $Core \left\langle X^{FRP}, S \right\rangle = Core^{FRP}(S)$. Since for $Core^{FRP}(S)$, coalitional deviations are not required to be FRP, it is obvious that $Core \left\langle X^{FRP}, S \right\rangle \supseteq Core^{FRP}(S)$ holds. To see the opposite direction, we only need to show that FRP condition is not binding for nested deviations. For this, notice that $a^*(T) \leq a^*(S)$ holds for all $T \subset S$. That is, $v_i(a^*(T \backslash \{i\})) < v_i(a^*(S \backslash \{i\}))$ holds for all $i \in T$, which implies that if $u_i \geq v_i(a^*(S \backslash \{i\}))$ then $u_i \geq v_i(a^*(T \backslash \{i\}))$ holds for all $i \in T$. This implies that all coalitional deviations that block $(S, a^*(S), u)$ must at least satisfy FRP condition. This proves our first claim, $Core \left\langle X^{FRP}, S \right\rangle = Core^{FRP}(S)$.

Now, let us consider non-nested coalitional deviations. Pick $(S, a^*(S), u) \in Core^{FRP}(S) = Core \left\langle X^{FRP}, S \right\rangle$. If for all nonnested T, $(S, a^*(S), u)$ is not blocked by any $(T, a^*(T), u') \in Core^{FRP}(T)$, then $(S, a^*(S), u) \in Core^{FRP}$ holds since $(S, a^*(S), u)$ is immune to nested deviations. Now, if $(S, a^*(S), u)$ is not blocked by all $(T, a^*(T), u') \in X^{FRP}(T)$, then $(S, a^*(S), u) \in Core \left\langle X^{FRP} \right\rangle$ holds. Since $Core^{FRP}(T) \subseteq X^{FRP}(T)$ for all T, for an allocation $(S, a^*(S), u)$ to be in $Core \left\langle X^{FRP} \right\rangle$, it needs to be immune to more deviations than to be in $Core^{FRP}(S)$. Thus, clearly, $Core^{FRP} \supseteq Core \left\langle X^{FRP} \right\rangle$ holds. We will show the opposite direction is also true in the following. Pick $(S, a^*(S), u) \in Core \left\langle X^{FRP}, S \right\rangle = Core^{FRP}(S) \subseteq Core^{FRP}$, and T is not nested from S. Suppose that there is $(T, a^*(T), u') \in X^{FRP}(T) \backslash Core^{FRP}(T)$ such that $u'_i > u_i$ for all $i \in T$. Then, there exists $T' \subset T$ such that $\sum_{i \in T'} u'_i < V(T')$. The FRP condition $u'_i \geq v_i(a^*(T' \backslash \{i\}))$ is trivially satisfied since $u'_i \geq v_i(a^*(T \backslash \{i\}))$ for all $i \in T'$ and $T \supset T'$. However, this implies that there exists $(T', a^*(T'), u'') \in X^{FRP}(T')$ with $u''_i > u'_i$ for all $i \in T'$ that blocks $(S, a^*(S), u) \in$

$Core\left\langle X^{FRP},S\right\rangle = Core^{FRP}(S)$. If $(T',a^*(T'),u'') \notin Core^{FRP}(T')$, then again there exists $T'' \subset T'$ with $(T'',a^*(T''),u''') \in X^{FRP}(T'')$ and $u_i''' > u_i''$ for all $i \in T''$ that blocks $(S,a^*(S),u) \in Core\left\langle X^{FRP},S\right\rangle = Core^{FRP}(S)$. This process must stop since $T \supset T' \supset T'' \supset$ ... and $X^{FRP}(\{i\}) = Core^{FRP}(\{i\}) = (\{i\},a^*(\{i\}),v_i(a^*(\{i\})-C(\{i\}),(v_j(a^*\{i\}))_{j\neq i})$. Thus, there exists $T''' \subset T$, and $(T''',a^*(T'''),u'''') \in Core^{FRP}(T''')$ that blocks $(S,a^*(S),u)$. This proves that whenever $(S,a^*(S),u)$ is blocked by non-nested T via $(T,a^*(T),u') \in X^{FTP}(T)$, it is also blocked by some $T''' \subset T$ via $(T''',a^*(T'''),u'''') \in Core^{FRP}(T''')$. This completes the proof of $Core^{FRP} = Core\left\langle X^{FRP}\right\rangle$. □

Proof of Proposition 3.

First, we construct a strategy profile σ, which will be shown to support $(S^*,a^*(S^*),u^*)$, where $u^* \in Core^{FRP}(S^*)$, as a PCPNE. In defining σ, we assign a CPNE utility profile to every subgame S'. Then, we show by way of contradiction that there is no credible and profitable deviation from σ.

A strategy profile in the second stage σ^2 is generated from utility allocations assigned in each subgame (we utilize truthful strategies that support utility outcomes). We partition the set of subgames $\mathcal{S} = \{S' \in 2^N : S' \neq \emptyset\}$ into three categories: $\mathcal{S}_1 = \{S^*\}$ on the equilibrium path, $\mathcal{S}_2 = \{S' \in \mathcal{S} : S' \cap S^* = \emptyset\}$, and $\mathcal{S}_3 = \{S' \in \mathcal{S}\backslash\mathcal{S}_1 : S' \cap S^* \neq \emptyset\}$. As Laussel and Le Breton (2001) show, a CPNE outcome in a subgame S' corresponds to a core allocation for S'. To support the equilibrium path $(S^*,a^*(S^*),u^*)$, we need to show that there is no credible deviation in the first stage. Since a credible deviation requires both free-riding-proofness and profitability, utility level $\bar{u}_i = \max\{u_i^*, v_i(S'\backslash\{i\})\}$ plays an important role as to whether or not player i joins a coalitional deviation.

We construct a core allocation for subgame S' with the algorithm described in Appendix A, starting with the initial value $\bar{u}$. Then we show that if there exists a credible deviation by coalition T, which induces $(S',a^*(S'),u')$ from $(S^*,a^*(S^*),u^*)$, then $(S \setminus S^*, a^*(S \setminus S^*), (u_i')_{i\in S'\setminus S^*}, (v_j(a^*(S' \setminus S^*)))_{j\notin S'\setminus S^*}) \in Core^{FRP}(S' \setminus S^*)$ and Pareto-dominates $(S^*,a^*(S^*),u^*)$. This is a contradiction to the presumption that $(S^*,a^*(S^*),u^*) \in Core^{FRP}$.

Thus, we will conclude that there is no credible deviation from $(S^*, a^*(S^*), u^*)$.

The construction of the core allocation for each subgame is as follows.

1. We assign $(S^*, a^*(S^*), u^*) \in Core^{FRP}$ to the on-equilibrium subgame S^*.

2. For any S' with $S' \cap S^* = \emptyset$, we assign an extreme point of the core for S' of a convex game. For an arbitrarily selected order ω over S', we assign payoff vector $u_{\omega(1)} = V(\{\omega(1)\}) - V(\emptyset)$, $u_{\omega(2)} = V(\{\omega(1), \omega(2)\}) - V(\{\omega(1)\})$, and so on, following Shapley (1971). Call this allocation $\hat{u}_{S'} \in Core(S', V)$ (see Property 1 in Appendix A).

3. For any S' with $S' \cap S^* \neq \emptyset$, we assign a core allocation in the following manner. It requires a few steps. First, we deal with the outsiders $S' \setminus S^*$. Let $\omega : \{1, \cdots, |S' \backslash S^*|\} \rightarrow S' \backslash S^*$ be an arbitrary bijection, and let $\hat{u}_{\omega(1)} = V(\{\omega(1)\})$, $\hat{u}_{\omega(2)} = V(\{\omega(1), \omega(2)\}) - V(\{\omega(1)\}), \cdots, \hat{u}_{\omega(|S' \backslash S^*|)} = V(S' \backslash S^*) - V((S' \backslash S^*) \backslash \{\omega(|S' \backslash S^*|)\})$. Such a core allocation minimizes the total payoffs for $S' \backslash S^*$ (Shapley, 1971). The rest $V(S') - V(S' \backslash S^*)$ goes to $S' \cap S^*$. Consider a reduced game of (S', V) on $S' \cap S^*$ with $u_{S' \backslash S^*}$ as given above and $\tilde{V}_{S' \cap S^*} : 2^{S' \cap S^*} \rightarrow \mathbb{R}$ such that $\tilde{V}_{S' \cap S^*}(Q) = V(Q \cup (S' \backslash S^*)) - \sum_{j \in S' \backslash S^*} u_j = V(Q \cup (S' \backslash S^*)) - V(S' \backslash S^*)$. By Property 2, we know that $u_{S' \cap S^*} \in Core(S' \cap S^*, \tilde{V}_{S' \cap S^*})$ if and only if $(u_{S' \cap S^*}, u_{S' \backslash S^*}) \in Core(S', V)$. For each $i \in S' \cap S^*$, let $\bar{u}_i = \max\{u_i^*, v_i(S' \backslash \{i\})\}$. By the algorithm in Appendix A, we construct a core allocation $\hat{u}_{S' \cap S^*}$ from vector $\bar{u}_{S' \cap S^*} = (\bar{u}_i)_{i \in S' \cap S^*}$ for the reduced game $\tilde{V}_{S' \cap S^*}$ of game $V : 2^{S'} \rightarrow \mathbb{R}$.

We support these core allocations by truthful strategies. Let $\sigma_i^1 = 1$ for $i \in S^*$, and $\sigma_i^1 = 0$ for $i \notin S^*$. Let $\sigma_i^2[S^*]$ be a truthful strategy relative to $a^*(S^*)$ such that $\sigma_i^2[S^*](a^*(S^*)) = v_i(a^*(S^*)) - u_i^*$ for all $i \in S^*$, and let $\sigma_i^2[S']$ be a truthful strategy relative to $a^*(S')$ with $\sigma_i^2[S'](a^*(S')) = v_i(a^*(S')) - \hat{u}_i(S')$ for all $i \in S'$. Since a core allocation with truthful strategies is assigned to every subgame, it is a CPNE. If there is a deviation from σ, therefore, it must happen in the first stage.

Suppose to the contrary that there exists a coalition T that profitably and credibly deviates from the equilibrium σ. Note that in the reduced game played by T, it must be a PCPNE deviation with σ'_T for given σ_{-T}. In the original equilibrium, S^* is the contribution group. This implies that every $i \in (N \backslash S^*) \backslash T$ plays $\sigma_i^1 = 0$, i.e., free-riding, in the first stage, while every $i \in S^* \backslash T$ plays $\sigma_i^1 = 1$ in the first stage and engages in the same strategy, i.e., the prescribed menu $\sigma_i^2(S')$ contingent to group S', in the second stage. Any $i \in T \backslash S^*$ has chosen $\sigma_i^1 = 0$ but chooses $\sigma_i^{1\prime} = 1$ upon deviation in the first stage, whereas $i \in T \cap S^*$ may or may not choose $\sigma_i^{1\prime} = 1$. Some may choose to free-ride by switching to 0, while others stay in the contribution group, adjusting their strategies in the second stage. To summarize, let S' be the contribution group formed as a result of T's deviation, i.e., $S' = S(\sigma_{-T}^1, \sigma_T^{1\prime})$. Then, there are five groups of players to be considered (see Figure 1).

(i) the members of $S^* \backslash S' \subset T$ that switch to free-riding after the deviation,

(ii) the members of $S' \backslash S^* \subset T$ that join the contribution group upon deviation,

(iii) the members of $(S^* \cap S') \backslash T \subset S'$ that still participate in the contribution group after the deviation, with the same prescribed menu in the second stage,

(iv) the members of $(S^* \cap S') \cap T \subset S'$ that change their strategies in the second stage,

(v) the members of $N \backslash (S' \cup S^*)$ that are outsiders both before and after the deviation.

Let the resulting allocation be $(S', a^*(S'), u')$. Since the deviation is profitable and credible, the members of T, i.e., those who are categorized in (i), (ii), and (iv), are better off after the deviation. That is,

$$
\begin{aligned}
v_i(a^*(S')) &\geq u_i^* \text{ for all } i \in S^* \backslash S', \\
u_i' &\geq \bar{u}_i \text{ for all } i \in S' \backslash S^*, \\
u_i' &\geq \bar{u}_i \text{ for all } i \in (S^* \cap S') \cap T,
\end{aligned}
$$

where $\bar{u}_i = \max\{u_i^*, v_i(a^*(S' \backslash \{i\}))\}$.

Given our supposition, the following claims must be true.

First we claim that members of (ii) exist and that $a^*(S') > a^*(S^*)$, as they are better off after the deviation. The set of players in (ii) is nonempty, since otherwise $S' \subset S^*$ and a coalitional deviation by T cannot be profitable, as $(S^*, a^*(S^*), u^*)$ is a core allocation. This result is from Proposition 1.

Claim 1. $S' \backslash S^* \neq \emptyset$ and $a^*(S') > a^*(S^*)$.

Since all players use truthful strategies in the strategy profile σ even after T's deviation, the members in (iii) (outsiders of T) obtain the same payoff vector $\hat{u}_{(S^* \cap S') \backslash T}(S')$ as in the original subgame CPNE for S'. This is because in subgame S' (even after deviation), $a^*(S')$ must be provided, as a CPNE (core) must be assigned to the subgame. Thus, we have the following for group (iii).

Claim 2. After the deviation by T, every $i \in (S^* \cap S') \backslash T \subset S'$ receives exactly $u'_i = \hat{u}_i$.

Since u' needs to be a CPNE payoff vector in the second stage of the reduced game by T, we have $\sum_{i \in S' \backslash S^*} u'_i \geq V(S' \backslash S^*)$ for $u_{S'}$ to be in $Core(S', V)$. By the construction of $\hat{u}_{S'}$, on the other hand, we have $\sum_{i \in S' \backslash S^*} \hat{u}_i = V(S' \backslash S^*)$. Thus, we have the following for group (ii).

Claim 3. $\sum_{i \in S' \backslash S^*} u'_i \geq V(S' \backslash S^*) = \sum_{i \in S' \backslash S^*} \hat{u}_i$.

The next claim shows that the counterpart of Claim 3 holds for group (iv).

Claim 4. $\sum_{i \in S' \cap S^* \cap T} u'_i = \sum_{i \in S' \cap S^* \cap T} \hat{u}_i$

Proof of Claim 4. Group (iv) consists of members of W, I, and L. Note that $u'_i \geq \bar{u}_i$ for any $i \in S' \cap S^* \cap T$ since otherwise they would have no incentive to join the deviation.

First consider the set W of winners in group (iv); we have $\hat{u}_i \geq \bar{u}_i$ by the definition of W. The contribution group S' must be immune to a coalitional deviation by W, so we have

$$\sum_{i \in W} u'_i \geq \tilde{V}(W) = \sum_{i \in W} \hat{u}_i,$$

where the equality holds by Lemma 2. As for players in I, we have $\hat{u}_i = \bar{u}_i$ by definition. Thus, it follows from $u'_i \geq \bar{u}_i$ that $u'_i \geq \hat{u}_i$ for any $i \in I$. Payoffs for losers, by definition, must satisfy $\hat{u}_i < \bar{u}_i$, so we have $u'_i > \hat{u}_i$ because $u'_i \geq \bar{u}_i$. However, it follows from Claim 2, Claim 3, and $\sum_{i \in S'} u'_i = \sum_{i \in S'} \hat{u}_i = V(S')$ that

$$\sum_{i \in S' \cap S^* \cap T} u'_i \leq \sum_{i \in S' \cap S^* \cap T} \hat{u}_i. \tag{3}$$

Together with $\sum_{i \in W} u'_i \geq \sum_{i \in W} \hat{u}_i$ and $\sum_{i \in I} u'_i \geq \sum_{i \in I} \hat{u}_i$, these imply that L is empty, and hence $\sum_{i \in S' \cap S^* \cap T} u'_i \geq \sum_{i \in S' \cap S^* \cap T} \hat{u}_i$. Consequently, we have from (3) that $\sum_{i \in S' \cap S^* \cap T} u'_i = \sum_{i \in S' \cap S^* \cap T} \hat{u}_i$. $\square$

Claims 2, 3, and 4 immediately imply the following for group (ii).

Claim 5. $\sum_{i \in S' \backslash S^*} u'_i = \sum_{i \in S' \backslash S^*} \hat{u}_i = V(S' \backslash S^*)$.

The final claim follows from Claim 5 and the supposition that the deviation by T is profitable and credible.

Claim 6. Consider a deviation by $S' \cup S^*$ such that $S' \setminus S^*$ is the resulting contribution group (all members in S^* stop contributing). Then the allocation $(S' \setminus S^*, a^*(S' \setminus S^*), (u'_i)_{i \in S' \backslash S^*}, (v_j(a^*(S' \backslash S^*)))_{j \notin S' \backslash S^*})$ is in $Core^{FRP}(S' \backslash S^*)$ and Pareto-dominates $(S^*, a^*(S^*), u^*)$.

Proof of Claim 6. Since the deviation by T is profitable, we have

$$\begin{aligned}
\sum_{i \in S' \backslash S^*} v_i(a^*(S' \backslash S^*)) - C(a^*(S' \backslash S^*)) &= V(S' \backslash S^*) \\
&= \sum_{i \in S' \backslash S^*} u'_i \\
&> \sum_{i \in S' \backslash S^*} v_i(a^*(S^*)).
\end{aligned}$$

Thus, we have $\sum_{i \in S' \backslash S^*} v_i(a^*(S' \setminus S^*)) > \sum_{i \in S' \backslash S^*} v_i(a^*(S^*))$, and hence $a^*(S' \setminus S^*) > a^*(S^*)$. Now, since the deviation by T is credible, and hence $u'_i \geq v_i(a^*(S' \setminus \{i\})) \geq v_i(a^*((S' \setminus S^*) \setminus \{i\}))$ for any $i \in S' \setminus S^*$, Claim 5 implies that $(S' \setminus S^*, a^*(S' \setminus S^*), (u'_i)_{i \in S' \backslash S^*}, (v_j(a^*(S' \setminus S^*)))_{j \notin S' \backslash S^*}) \in Core^{FRP}(S' \setminus S^*)$.

Next, we show that $((u'_i)_{i\in S'\setminus S^*}, (v_j(a^*(S'\setminus S^*)))_{j\notin S'\setminus S^*})$ Pareto-dominates u^*. First, the profitability of the deviation by T immediately implies that $u'_i \geq v_i(a^*(S^*)) = u^*_i$ for any $i \in S'\setminus S^*$. Thus, we have shown the Pareto-domination for group (ii). Pareto-domination for group (v) is immediate from $a^*(S'\setminus S^*) > a^*(S^*)$. As for groups (i), (iii), and (iv), i.e., for all $i \in S^*$, we first note that since $u^* \in Core(S^*)$ and the game V is convex, we have $u^*_i \leq V(S^*) - V(S^*\backslash\{i\})$ (Shapley, 1971). Now,

$$\begin{aligned}
&V(S^*) - V(S^*\backslash\{i\}) \\
&= \sum_{j\in S^*} v_j(a^*(S^*)) - C(a^*(S^*)) - \left(\sum_{j\in S^*\backslash\{i\}} v_j(a^*(S^*\backslash\{i\})) - C(a^*(S^*\backslash\{i\}))\right) \\
&< v_i(a^*(S^*)) \\
&\quad + \sum_{j\in S^*\backslash\{i\}} v_j(a^*(S^*)) - C(a^*(S^*)) - \left(\sum_{j\in S^*\backslash\{i\}} v_j(a^*(S^*\backslash\{i\})) - C(a^*(S^*\backslash\{i\}))\right) \\
&< v_i(a^*(S'\backslash S^*)),
\end{aligned}$$

where the last inequality holds since $\sum_{j\in S^*\backslash\{i\}} v_j(a) - C(a)$ is maximized at $a = a^*(S^*\backslash\{i\})$. This proves that all members of groups (i), (iii), and (iv) are better off in the allocation $(S'\backslash S^*, a^*(S'\backslash S^*), (u'_i)_{i\in S'\backslash S^*}, (v_j(a^*(S'\backslash S^*)))_{j\notin S'\backslash S^*})$. Hence, we conclude that $(S^*, a^*(S^*), u^*) \in Core^{FRP}$ is Pareto-dominated by $(S'\backslash S^*, a^*(S'\backslash S^*), (u'_i)_{i\in S'\backslash S^*}, (v_j(a^*(S'\backslash S^*)))_{j\notin S'\backslash S^*})$, which is in $Core^{FRP}(S'\backslash S^*)$. □

The statement of Claim 6 is an apparent contradiction to $(S^*, a^*(S^*), u^*) \in Core^{FRP}$ (see Proposition 2). Thus, we have shown that there is no profitable and credible deviation from the constructed strategy profile σ, so σ is a PCPNE. □

Proof of Proposition 4

Suppose to the contrary that for any natural number l, there exists $r \geq l$ such that $(S_r, a^*(S_r), u^*_r) \in Core^{FRP}(V^r)$ and $a^*(S_r) \geq \bar{a}$. This implies that there exists an increasing sequence of natural numbers r that satisfy $(S_r, a^*(S_r), u^*_r) \in Core^{FRP}(V^r)$. We show that (under this supposition) for any r with $(S_r, a^*(S_r), u^*_r) \in Core^{FRP}(V^r)$ and any $i_q \in S_r$,

$a^*(S_r \setminus \{i_q\})$ approaches $a^*(S_r)$ as $r \to \infty$, and hence the left-hand side of

$$\sum_{i \in N} \frac{m_i(S)}{r} [v_i(a^*(S)) - v_i(a^*(S \setminus \{i_q\}))] \geq C(a^*(S)). \tag{4}$$

diminishes to zero (since $v_i'(a^*(S)) \leq v_i'(\bar{a}) < \infty$). Since $C(a^*(S)) \geq C(\bar{a}) > 0$, this implies that (1) is violated eventually as $r \to \infty$, which in turn leads to a contradiction to $(S_r, a^*(S_r), u_r^*) \in Core^{FRP}(V^r)$.

Now, $a^*(S_r)$, the public goods provision level induced by the contribution group S_r, is chosen so as to satisfy the first-order condition:

$$\sum_{j \in N} \frac{m_j(S_r)}{r} v_j'(a^*(S_r)) - C'(a^*(S_r)) = 0, \tag{5}$$

where $[m_j(S_r)/r]v_j'(a) = \sum_{i_q \in S} v_{i_q}^{r\prime}(a)$. For any r, the left-hand side of (5) is continuous and strictly decreasing in the public goods provision level a since $v_j'' \leq 0$ and $C'' > 0$ (as Figure 2 illustrates). Similarly, for any $i_q \in S_r$, the optimality of public goods provision requires that $a^*(S_r \setminus \{i_q\})$ satisfy

$$\sum_{j \in N} \frac{m_j(S_r \setminus \{i_q\})}{r} v_j'(a^*(S_r \setminus \{i_q\})) - C'(a^*(S_r \setminus \{i_q\})) = 0, \tag{6}$$

or equivalently

$$\sum_{j \in N} \frac{m_j(S_r)}{r} v_j'(a^*(S_r \setminus \{i_q\})) - \frac{v_i'(a^*(S_r \setminus \{i_q\}))}{r} - C'(a^*(S_r \setminus \{i_q\})) = 0,$$

where the second term in the second equation represents the free-rider i_q's marginal benefit from the public goods provision.

Now, we claim that for any $\epsilon \in (0, \bar{a})$, there exists a positive integer r_ϵ such that for any $r \geq r_\epsilon$,

$$\sum_{j \in N} \frac{m_j(S_r)}{r} v_j'(a^*(S_r) - \epsilon) - \frac{v_i'(a^*(S_r) - \epsilon)}{r} - C'(a^*(S_r) - \epsilon) > 0,$$

i.e., the left-hand side of (6), evaluated at $a = a^*(S_r) - \epsilon$ instead of $a^*(S_r \setminus \{i_q\})$, is positive as Figure 2 shows. Together with $v_j'' \leq 0$ and $C'' > 0$, this implies that $a^*(S_r \setminus \{i_q\}) \in (a^*(S_r) - \epsilon, a^*(S_r))$, which in turn implies the convergence of $a^*(S_r \setminus \{i_q\})$ to $a^*(S_r)$.

To show the claim, we first define the minimum C'' over the relevant range as $c \equiv \min_{a \in [0, a^*(N)]} C''(a)$. It follows from $C'' > 0$ that $c > 0$. Now, for any r, it follows from (5) and Taylor's formula that there exists $a' \in [a^*(S_r) - \epsilon, a^*(S_r)]$ such that

$$\begin{aligned}
&\sum_{j \in N} \frac{m_j(S_r)}{r} v_j'(a^*(S_r) - \epsilon) - C'(a^*(S_r) - \epsilon) \\
&= \sum_{j \in N} \frac{m_j(S_r)}{r} v_j'(a^*(S_r)) - C'(a^*(S_r)) \\
&\quad - \epsilon \left[\sum_{j \in N} \frac{m_j(S_r)}{r} v_j''(a') - C''(a') \right] \\
&= \epsilon \left[C''(a') - \sum_{j \in N} \frac{m_j(S_r)}{r} v_j''(a') \right] \\
&\geq c\epsilon,
\end{aligned}$$

where we have used $v_j'' \leq 0$ to derive the last inequality. On the other hand, it follows from $v_i'(a^*(S_r) - \epsilon) \leq v_i'(\bar{a} - \epsilon)$ (as $a^*(S_r) > \bar{a}$) that there exists r_ϵ such that

$$\frac{v_i'(a^*(S_r) - \epsilon)}{r} \leq \frac{v_i'(\bar{a} - \epsilon)}{r} < \frac{c\epsilon}{2}$$

holds for any $r \geq r_\epsilon$. Then the claim follows immediately since

$$\begin{aligned}
\sum_{j \in N} \frac{m_j(S_r)}{r} v_j'(a^*(S_r) - \epsilon) - \frac{v_i'(a^*(S_r) - \epsilon)}{r} - C'(a^*(S_r) - \epsilon) &> c\epsilon - \frac{c\epsilon}{2} \\
&> 0.
\end{aligned}$$

Now, we have from $m_i(S_r) \leq r$ and the claim established above that

$$\begin{aligned}
&\sum_{i \in N} \frac{m_i(S_r)}{r} \left[v_i(a^*(S_r)) - v_i(a^*(S_r \setminus \{i_q\})) \right] \\
&\leq \sum_{i \in N} \left[v_i(a^*(S_r)) - v_i(a^*(S_r \setminus \{i_q\})) \right] \to 0 \text{ as } r \to \infty.
\end{aligned}$$

Since $C(a^*(S)) > C(\bar{a}) > 0$, we have shown that there exists $\bar{r}(\bar{a})$ such that for any $r \geq \bar{r}(\bar{a})$, the free-riding-proofness condition (1) fails to be satisfied, which implies that $a^*(S^*) < \bar{a}$ for any $(S^*, a^*(S^*), u^*) \in Core^{FRP}(V^r)$ when $r \geq \bar{r}(\bar{a})$.

Appendix C

In this appendix, we provide some logical relationships between PCPNE of our game and other equilibrium concepts of (possibly other) noncooperative coalition formation games. For the sake of simplicity, we assume that the TU game generated from the public goods provision game is not only a convex game but also a *strictly* convex game (i.e., inequalities are strict). This is not at all a restrictive assumption when the public goods space is the nonnegative real line.

First note that even within our game, PCPNE is different from a subgame perfect Nash equilibrium (SPNE) with a coalition-proof Nash equilibrium (CPNE) assigned to each subgame (i.e., a Nash equilibrium in the participation game with a CPNE assigned to each subgame of the contribution stage). In PCPNE, when a group of players deviates in the first stage, they can coordinate their strategies in the following subgames in order to support a target outcome (as long as such a deviation strategy profile is credible in the recursive sense). In contrast, in SPNE, only a single player can change her strategy in the first stage and also in the following subgame. Since an equilibrium strategy is taken in each subgame, the deviating player has no incentive to switch her strategies in the subgames. Thus, given that a CPNE is assigned to every subgame, we need only check if there exists a unilateral deviation incentive in the first stage to see if a given strategy profile for the entire game is SPNE. This distinction makes a big difference. In the following, we will compare the set of equilibrium outcomes of various rules of games including (noncooperative) coalition bargaining games.

In this paper, we have analyzed $Core^{FRP}(S)$ for $S \subseteq N$, $Core^{FRP}$, and $PCPNE$ of our voluntary-participation game with common agency games as the second-stage subgames. Recall that we have found that PCPNE of our game is equivalent to $Core^{FRP}$ and $Core\langle X^{FRP}\rangle$. Here, we consider the following other possible games and equilibrium concepts:

1. SPNE with CPNE of the common agency game, i.e., Nash equilibrium (NE) of our voluntary-participation game with CPNE assigned to each common agency subgame.

2. SPNE with the strong Nash equilibrium (SNE) of the common agency game, i.e., NE of our voluntary-participation game with SNE assigned to each common agency subgame.

3. Perfect strong Nash equilibrium (PSNE), i.e., SNE of our voluntary-participation game with SNE assigned to each common agency subgame.

4. SPNE with the stationary subgame perfect Nash equilibrium (SSPNE) of Perry and Reny's (1994) or Moldovanu and Winter's (1995) noncooperative coalition bargaining games in the second stage that follows the voluntary participation game in the first stage.

We restrict our attention to the open-membership participation game in the first stage since voluntary participation in the common agency contribution game is of primary interest to us. We will show the logical relationship that is schematically described in Figure 3.

Claim 1. Games 1, 2, and 4 generate a common set of outcomes that includes $PCPNE = Core^{FRP}$ of our game. This inclusion relationship is strict in general.

Proof. In our common agency (sub)game, the set of CPNE outcomes, the set of SNE outcomes, and the core of the TU game generated from our public good economy are all equivalent. Perry and Reny (1994) show that the core of a TU game is implementable by a noncooperative coalitional bargaining game when it is totally balanced. Similarly, Moldovanu and Winter (1995) show that the core of an NTU game is implementable by their noncooperative coalitional bargaining game when the NTU game has a nonempty core in all subgames. Thus, in the subgame $S \subseteq N$ of games 1, 2, and 4 generate $Core(S)$. The sets of outcomes of these three games are the same, since the Nash equilibrium concept is adopted in the first stage voluntary-participation game of all the three.

As for the comparison with $PCPNE = Core^{FRP}$, it is clear that every PCPNE allocation is included in the outcome allocations of the above three games, since the perfectness of PCPNE is more demanding than Nash equilibrium.

We show that this inclusion relationship is strict with the example in the main text. Since the TU game generated from the public goods provision problem is convex, we can utilize Property 1 of Appendix A (Shapley, 1972). That is, the worst core allocation for each player i is $V(\{i\})$ in any convex games. First, consider singleton coalition $S = \{11\}$ as the contribution group. In this case, $a^*(\{11\}) = 11$ and $V(\{11\}) = 60.5$. The unique allocation of $Core(S)$ is $(S, a^*(S), u) = (\{11\}, 11, (60.5, 55, 33, 11))$. Although it does not belong to $Core^{FRP}$ (or the set of allocations in PCPNE), we can support it as a subgame perfect Nash equilibrium of games 1, 2, and 4. Since the participation stage just demands being a Nash equilibrium in these three games, we only need to consider three subgames $S' = \{11, 1\}$, $S'' = \{11, 3\}$, and $S''' = \{11, 5\}$, which yield $V(S') = 72$, $V(S'') = 98$, and $V(S''') = 128$, respectively. We assign an extreme point of the core to each of subgames S', S'', and S''': $(u'_{11}, u'_1) = (71.5, 0.5)$, $(u''_{11}, u''_3) = (93.5, 4.5)$, and $(u'''_{11}, u'''_5) = (115.5, 12.5)$, respectively. Clearly, none of 1, 3, and 5 has an incentive to deviate. Thus, the allocation $(S, a^*(S), u) = (\{11\}, 11, (60.5, 55, 33, 11))$ is in SPNE of games 1, 2, and 4. □

Claim 2. Game 3 generates the set of outcomes that is included in $PCPNE = Core^{FRP}$ of our game. Moreover, it is empty unless there is a grand coalition free-riding-proof core allocation.

Proof. Clearly, $PSNE \subset PCPNE$ since $SNE \subset CPNE$. In the public goods provision problem, for all $S \subsetneq N$ and all $(S, a^*(S), u) \in Core(S)$, there is an allocation $(N, a^*(N), u') \in Core(N)$ with $u' > u$; the grand coalition with this allocation blocks the allocation $(S, a^*(S), u)$. Thus, unless there is an FRP-Core allocation with the grand coalition, PSNE is an empty set. □

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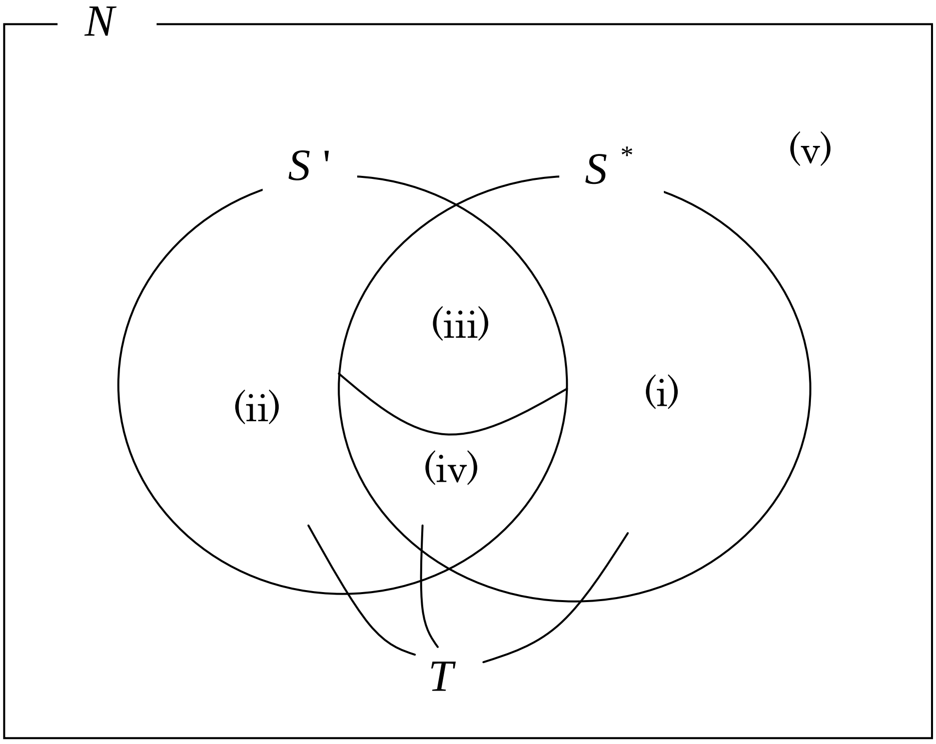

T: Deviating coalition (i) + (ii) + (iv)

S^*: Equilibrium lobby

S': Off-equilibrium lobby

Figure 1. A Deviation from S^*

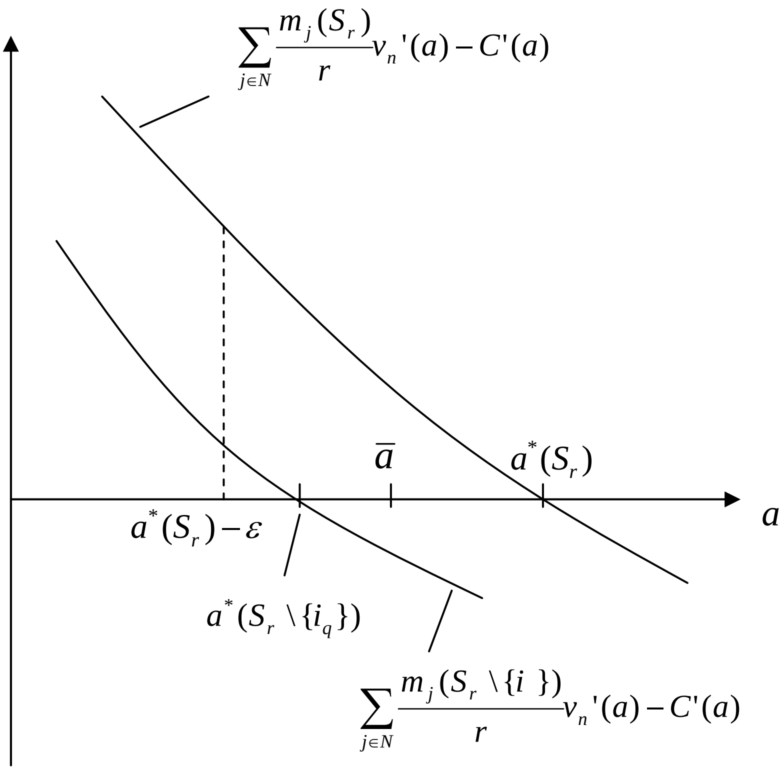

Figure 2. Convergence of $a^*(S_r\setminus\{i_q\})$ to $a^*(S_r)$

PSNE 1. SNE 2. SNE	$\subseteq$	$PCPNE = Core^{FRP}$	$\subseteq$	SPNE with CPNE 1. NE 2. CPNE	=	SPNE with SNE 1. NE 2. SNE	=	SPNE (noncooperative) 1. NE 2. SSPNE of the noncooperative coalitional bargaining games

Figure 3. Relationships with other equilibrium concepts