Games of School Choice under the Boston Mechanism

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Abstract

Many school districts in the U.S. use a student assignment mechanism that we refer to as the Boston mechanism. Under this mechanism a student loses his priority at a school unless his parents rank it as their first choice. Therefore parents are given incentives to rank high on their list the schools where the student has a good chance of getting in. We characterize the Nash equilibria of the induced preference revelation game. An important policy implication of our result is that a transition from the Boston mechanism to the student-optimal stable mechanism would lead to unambiguous efficiency gains. (*JEL* C78, D61, D78, I20)

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1 Introduction

In the U.S., many school choice programs that assign children to public schools rely on the centralized student assignment mechanism that is currently used in Boston. Other major school districts that use versions of this mechanism include Cambridge,Charlotte, Denver, Minnesota, Seattle and St.Petersburg-Tampa. Under the *Boston mechanism* a student who is not assigned to his top ranked school A is considered for his second choice B only after the students who have top-ranked B. Therefore a student loses his priority at a school unless his family ranks it as their first choice. In particular it is typically not in the best interest of parents to reveal their true preferences.[1] Such preference manipulation is often advocated in local press. Consider the following statement from the Seattle Press:[2]

> The method the school district uses to sort the school choice requests gives first priority to students who are already enrolled at that school. Next in line come those students with siblings at the school. Both of these factors are beyond your control. These students are sure things. Enrollment at these schools is theirs for the asking. No amount of strategizing, short of polling all existing students to determine how many have younger siblings about to enter the school, can help you here. Third in line, and the first effect of any real choice, are those students who live in the school's reference area. This is why you have such an excellent chance of getting into your reference school if you make it your top choice. Choosing another neighborhood's reference school, however, puts a lot of kids in line ahead of yours. That reduces your chances of getting in, particularly if the school has small classes.

To see how such preference misrepresentation may lead to an efficiency loss, consider three schools A, B, and C each having 100 students in its reference area and a class size 100. Let us assume for simplicity that the only priority taken into account is proximity, that is, students in a given school's reference area are given priority for that school; and a lottery number is used to break ties. Suppose that school C is the least desired school from the perspective of every family and in each reference area, 50 families prefer A over B and the other 50 prefer B over A.

Consider a student i from the reference area A whose parents prefer school B to school A. If they rank B as their first choice then she loses her priority at A, hence it becomes difficult for i to get a seat at her reference area school A if she can not get a seat at school B. Hence by top ranking their true favorite B, i's parents risk their child to be assigned to their least preferred

[1] Chen and Sönmez [2003] report an experiment in which about 80 percent of the subjects mis-represent their preferences under the Boston mechanism. In their experimental setting the mis-representation rate increases among subjects who think there will be stiff competition for their first choice.

[2] Charles Mas, *Navigating the School Choice Maze*, The Seattle Press, December 30, 1998, http://www.seattlepress.com/article-483.html, accessed 02/10/2004.

school C. Alternatively i's parents may adopt the safer strategy and ensure i a seat at school A by ranking A as their first choice.

As more students from the B area submit B as their first choice, it becomes more difficult for i to get a seat at B when her parents rank B as their first choice, hence the safer strategy becomes more attractive. The situation is completely symmetric for a student j from the reference area B whose parents prefer school A to school B. The safer strategy for j's parents is to rank B as their top choice, and this strategy becomes more attractive as more students from the A area top rank A. As a result, it is an equilibrium under the Boston mechanism for each family to play their safe strategy and top rank the school in their reference area.[3] Under this equilibrium, every student is assigned to his/her reference area school. Note that it is feasible under the given level of resources, to assign all the students from the reference area of A who prefer B to A, to school B; and all the students from the reference area of B who prefer A to B, to school A. Such a reallocation of seats would improve the welfare of 100 families without affecting the others, illustrating the aggregate efficiency loss under the Boston mechanism.

In this paper, we characterize the extent of the efficiency loss suggested by the above example and identify the part of the inefficiency that can be recovered without violating the priorities. To understand how families choose to distort their rankings *in equilibrium*, we will identify the Nash equilibria of the preference revelation game induced by the Boston mechanism. In order to describe the set of Nash equilibrium outcomes, we shall connect school choice with an important model which has played a prominent role in the mechanism design literature. The school choice model (Abdulkadiroğlu and Sönmez [2003]) is closely related to the well-known *two-sided matching markets* (Gale and Shapley [1962]). The key difference between the two models is that in the former schools are indivisible objects which shall be assigned to students based on student preferences and school priorities whereas in the latter parties in both sides of the market are agents who have preferences over the other side and whose welfare are taken into consideration. While school priorities are determined by the school district based on state/local laws (and/or education policies) and do not necessarily represent school tastes, one can formally treat school priorities as school preferences and hence obtain a two-sided matching market for each school choice problem (see Abdulkadiroğlu and Sönmez [2003], Balinski and Sönmez [1999], and Ergin [2002]). Consequently concepts/findings in two-sided matching have their counterparts in school choice.

The central notion in two-sided matching is *stability*. Importance of this concept does not diminish in the context of school choice because if a matching is not stable then there is a student-school pair (i, s) such that (1) student i prefers school s to his assignment and (2) either school s has some empty seats or student i has higher priority than another student who is assigned a seat at school s. In either case student i can seek legal action against the school district for not assigning

[3] Note that this is one of many equilibria.

him a seat at school s. It is well-known that there exists a stable matching and furthermore there exists a stable matching which is preferred by any student to any other stable matching (Gale and Shapley [1962]). This matching is known as the *student-optimal stable matching* and it has played a key role in the re-design of U.S. hospital-intern market in 1998 (see Roth [2002], Roth and Peranson [1999]). The student-optimal stable matching can be obtained in several steps with the following *student-proposing deferred acceptance algorithm*: At Step 1 each student "proposes" to her first choice and each school tentatively assigns its seats to its proposers one at a time in their priority order. Any remaining proposers are rejected at the end of Step 1. In each of the following steps (a) each student who was rejected in the previous step proposes to her next choice if one remains, and (b) each school considers the students it has been holding together with its new proposers, tentatively assigns its seats to these students one at a time in priority order and rejects the remaining proposers. The algorithm terminates when no student proposal is rejected, and each student is assigned her final tentative assignment. Besides the fact that it is the most efficient stable mechanism, another desirable feature of the *student-optimal stable mechanism* is that under this mechanism it is a dominant strategy for student families to state their true rankings of the schools (Dubins and Freedman [1981], Roth [1982]).

While the student-optimal stable mechanism is well analyzed, not much is known about the Boston mechanism despite its widespread use at many school districts. In our main result we describe the Nash equilibrium outcomes of the preference revelation game induced by the Boston mechanism: *The set of Nash equilibrium outcomes is equal to the set of stable matchings under the true preferences*. This result allows us to make welfare comparison between the student-optimal stable mechanism and the Boston mechanism: The preference revelation game induced by the student-optimal stable mechanism has a dominant strategy equilibrium (which is truthful-revelation) and its outcome is either equal to or Pareto dominates the Nash equilibrium outcomes of the Boston mechanism. In that sense the outcome of the student-optimal stable mechanism is the best one can hope for under the Boston mechanism. An important policy implication is that a transition to student-optimal stable mechanism may result in significant efficiency gains in Boston, Cambridge, Charlotte, Denver, Minneapolis, Seattle, St. Petersburg-Tampa, and other districts which rely on variants of the Boston mechanism.[4] Our main result is fairly robust in a number of directions and our characterization extends:

1. to the case with strategic schools when *Nash equilibria in undominated strategies* is considered,

[4]After the first version of this paper was written, officials at the Boston Public Schools have authorized a study concerning an empirical analysis of the Boston mechanism and a possible transition to the student-optimal stable mechanism (Abdulkadiroğlu, Pathak, Roth and Sönmez [2005]). Roughly around the same time New York City Department of Education adopted a version of the student-optimal stable mechanism for the assignment of more than 90,000 eighth graders to public highschools (Abdulkadiroğlu, Pathak and Roth [2005]).

2. to the the case where there are capacity constraints on various types of students, and
3. to the more general class of priority matching mechanisms (Roth [1991]) when students are allowed to veto any subset of schools.

In addition to its policy implications, our paper also contributes to the theory of implementation in matching markets.[5] There are a number of papers that analyze equilibria induced by various mechanisms in the context of *marriage problems* (i.e. two-sided matching markets where each agent has only one slot.) One important negative result in this context is that preference revelation games induced by stable mechanisms may have Nash equilibria with unstable outcomes (Alcalde [1996]), and indeed given any Pareto efficient and individually rational mechanism the set of Nash equilibrium outcomes of the induced preference revelation game is the set of individually rational matchings (Sönmez [1997]). If, however, a refinement of Nash equilibria that allows pairs (one from each side of the market) is considered as the underlying equilibrium concept, then the set of equilibrium outcomes of these games is the set of stable matchings (Ma [1995], Shin and Suh [1996], Sönmez [1997]).[6] Our main result shows that the negative result mentioned above is avoided when only one side of the market is strategic: There exists a Pareto efficient and individually rational mechanism (the Boston mechanism) where the set of Nash equilibrium outcomes of the induced preference game is the set of stable matchings.

The fact that student families have incentives to misrepresent their preferences under the Boston mechanism was first brought into the attention of economists by Abdulkadiroğlu and Sönmez [2003]. They also noted that the outcome of the Boston mechanism may be unstable under the stated preferences and is therefore vulnerable to legal action by unsatisfied students and their parents. Our result shows that although the Boston mechanism is not stable, its equilibrium outcomes are stable with respect to true preferences. In particular, in equilibrium no family can ensure their child a seat in a more preferred school through legal action, hence do not have any incentives to initiate a lawsuit.

2 School Choice and Two-Sided Matching

In a **school choice problem** (Abdulkadiroğlu and Sönmez [2003]) there are a number of students each of whom should be assigned a seat at one of a number of schools. Each student has strict

[5] We can restate our result using implementation theory jargon as follows: The Boston mechanism implements the stable correspondence in Nash equilibria.

[6] See also Alcalde and Romero-Medina [2000], Kara and Sönmez [1996, 1997], Konishi and Ünver [2003], Shinotsuka and Takamiya [2003], Sotomayor [2003], Tadenuma and Toda [1998], Teo, Sethuraman and Tan [2001] for additional results on implementation in two-sided matching markets and Jackson [2001] for a recent and comprehensive survey on implementation theory.

preferences over all schools and each school has a strict priority ranking of all students. Each school has a maximum capacity but there is no shortage of the total number of seats.

Formally a school choice problem consists of:

1. a set of students $I = \{i_1, \ldots, i_n\}$,
2. a set of schools $S = \{s_1, \ldots, s_m\}$,
3. a capacity vector $q = (q_{s_1}, \ldots, q_{s_m})$,
4. a list of strict student preferences $P_I = (P_{i_1}, \ldots, P_{i_n})$, and
5. a list of strict school priorities $f = (f_{s_1}, \ldots, f_{s_m})$.

Here sP_is' means that student i strictly prefers school s to school s', q_s denotes the capacity of school s where $\Sigma_{s \in S}\, q_s \geq |I|$, and f_s denotes the strict priority ordering of students at school s.

The school choice problem is closely related to the well-known **two-sided matching markets** (Gale and Shapley [1962]).[7] Two-sided matching markets have been extensively studied and successfully applied in the American and British entry-level labor markets (see Roth [1984, 1991]). The key difference between the two models is that in school choice schools are "objects" to be consumed by the students whereas in two-sided matching participants in both sides of the market are agents who have preferences over the other side.

Formally a two-sided matching market consists of:

1. a set of students $I = \{i_1, \ldots, i_n\}$,
2. a set of schools $S = \{s_1, \ldots, s_m\}$,
3. a capacity vector $q = (q_{s_1}, \ldots, q_{s_m})$,
4. a list of strict student preferences $P_I = (P_{i_1}, \ldots, P_{i_n})$, and
5. a list of strict school preferences $P_S = (P_{s_1}, \ldots, P_{s_m})$.

Here P_s denotes the strict preference relation of school s over all students.

The two-sided matching theory have immediate implications on school choice. That is because, school priorities in the context of school choice can be interpreted as school preferences in the context of college admissions (see Abdulkadiroğlu and Sönmez [2003], Balinski and Sönmez [1999], Ehlers and Klaus [2004], Ergin [2002], and Kesten [2004]).

The outcome of both school choice problems and two-sided matching markets is known as a **matching**. Formally a matching $\mu : I \longrightarrow S$ is a function from the set of students to the set of

[7]Throughout the paper we consider the *many-to-one* version of two-sided matching markets. These problems are also knows as *college admissions problems*.

schools such that no school is assigned to more students than its capacity. Let $\mu(i)$ denote the assignment of student i under matching μ. Note that $\mu^{-1}(s)$ is the set of students each of whom is matched to school s under matching μ.

In the two-sided matching context, a student-school pair (i, s) is said to **block** a matching μ if either (1) student i prefers school s to its assignment $\mu(i)$ and school s has empty seats under μ, or (2) student i prefers school s to its assignment $\mu(i)$ and school s prefers student i to at least one of the students in $\mu^{-1}(s)$. A matching is **stable** if and only if there is no student-school pair that blocks it. Stability has been central to the two-sided matching literature. It is by now well known that not only the set of stable matchings is non-empty for each two-sided matching market, but also there exists a stable matching which is at least as good as any stable matching for any student (Gale and Shapley [1962]). This matching is known as the **student-optimal stable matching**.

Given a school choice problem we refer to a matching to be **stable** whenever it is stable for the induced two-sided matching market that is obtained by interpreting school priorities as school preferences. We refer to the mechanism that selects the student-optimal stable matching for each school choice problem as the **student-optimal stable mechanism**. By definition the student-optimal stable mechanism always yields a matching that is at least as good as any stable matching for any student. Moreover it is strategy-proof, that is truthful-preference revelation is always in students' best interest (Dubins and Freedman [1981], Roth [1982]).

3 The Boston Student Assignment Mechanism

A **student assignment mechanism** is a systematic procedure that selects a matching for each school choice problem. The following mechanism is the most widely used student assignment mechanism in real-life applications of school choice problems.[8]

The Boston Mechanism: For each school a strict priority ordering of students is determined, each student submits a preference ranking of the schools, and the key phase is the choice of a matching based on fixed priorities and submitted preferences.

Round 1: In Round 1 only the first choices of the students are considered. For each school, consider the students who have listed it as their first choice and assign seats of the school to these students one at a time following their priority order until either there are no seats left or there is no student left who has listed it as his first choice.

Round 2: Consider the remaining students. In Round 2 only the 2$^{\text{nd}}$ choices of these students are considered. For each school with still available seats, consider the students who have listed it as their 2$^{\text{nd}}$ choice and assign the remaining seats to these students one at a time following their

[8] See *Introducing the Boston Public Schools 2002*, http://boston.k12.ma.us/teach/assign.asp, accessed 02/10/2004.

priority order until either there are no seats left or there is no student left who has listed it as his 2^{nd} choice.

In general, at

Round k: Consider the remaining students. In Round k only the k^{th} choices of these students are considered. For each school with still available seats, consider the students who have listed it as their k^{th} choice and assign the remaining seats to these students one at a time following their priority order until either there are no seats left or there is no student left who has listed it as his k^{th} choice.

The procedure terminates when each student is assigned a seat at a school.

We next present a simple example which illustrates the working of the Boston mechanism.

Example 1: Let $I = \{i_1, i_2, i_3, i_4, i_5, i_6\}$ be the set of students, $S = \{a, b, c, d\}$ be the set of schools, and $q = (2, 2, 1, 1)$ be the school capacity vector. Student priorities at schools as well as their preferences are as follows:

$$\begin{array}{ll}
 & P_{i_1} : a \ldots \\
f_a : i_5 - i_1 - i_2 - i_3 \ldots & P_{i_2} : a \ldots \\
f_b : i_5 - i_6 - i_3 \ldots & P_{i_3} : a - b \ldots \\
f_c : i_4 - i_5 - i_6 \ldots & P_{i_4} : c \ldots \\
f_d : i_5 - i_6 \ldots & P_{i_5} : c - a - b - d \\
 & P_{i_6} : c - a - b - d
\end{array}$$

Round 1: Only the first choices of students are considered and those with higher priorities are accomodated. Each of students i_1 and i_2 is assigned a seat at school a; i_4 is assigned a seat at school c. At the end of Round 1, b has 2 and d has 1 seat available; students i_3, i_5, and i_6 are unassigned.

Round 2: Remaining students are considered for their second choices. There is no seat left at school a so students i_5, i_6 will not be accomodated in this round (too bad for student i_5 who lost the highest priority at school a) and student i_3 is assigned at seat at school b. Therefore at the end of Round 2, each of schools b, d has 1 seat available and students i_5, i_6 are unassigned.

Round 3: Remaining students are considered for their third choices and student i_5 is assigned a seat at school b. At the end of Round 3, school d has 1 seat available and student i_6 is unassigned.

Round 4: The only remaining student i_6 is assigned a seat at his forth choice school d.

Therefore the outcome of the Boston mechanism is:

$$\begin{pmatrix} i_1 & i_2 & i_3 & i_4 & i_5 & i_6 \\ a & a & b & c & b & d \end{pmatrix}.$$

The Boston mechanism is a special case of a **priority matching mechanism** (Roth [1991]) versions of which had been used to match medical school graduates (interns) to supervising consultants in several regions of UK starting with late 1960's. Each of these priority matching mechanisms were subsequently abandoned from the UK hospital-intern markets. Consider a school choice problem with n students and m schools. Under the Boston mechanism any student-school pair that rank each other first has the highest **match priority**. Roth [1991] refers to any such match as a **$(1,1)$ match**. Similarly define a **(k,ℓ) match** to be a match between a pair such that the student ranks the school k^{th} in his preferences and he has the ℓ^{th} priority at the school. The Boston mechanism first forms any feasible $(1,1)$ match, next any feasible $(1,2)$ match, ..., next any feasible $(1,n)$ match, next any feasible $(2,1)$ match, next any feasible $(2,2)$ match, ..., next any feasible $(2,n)$ match, next any feasible $(3,1)$ match, ..., and last in hierarchy is any feasible (m,n) match. A priority matching mechanism is a generalization of this idea but it can differ in the match priority hierarchy. Note that the match priority is lexicographic under the Boston mechanism: It first considers the student preferences and only then the school priorities. A similar lexicographic priority matching mechanism was used in Edinburgh in 1967 and 1968.

Roth [1991] shows that no priority matching mechanism is stable and the Boston mechanism is no exception. In particular a student may lose his priority at a school unless he ranks it as his first choice and hence truthful preference revelation may not be in students' best interest.[9] Students and their families are forced to play a preference revelation game that we will analyze in the next section. As field evidence, preference manipulation is often advocated by the local press. In addition to the Seattle Press story quoted in the Introduction, consider the following statement from a recent story in the St.Petersburg Times:[10]

> Make a realistic, informed selection on the school you list as your first choice. It's the cleanest shot you will get at a school, but if you aim too high you might miss.
>
> Here's why: If the random computer selection rejects your first choice, your chances of getting your second choice school are greatly diminished. That's because you then fall in line behind everyone who wanted your second choice school as their first choice. You can fall even farther back in line as you get bumped down to your third, fourth and fifth choices.

Apparently many parents are well aware of the vulnerability of the Boston mechanism to preference manipulation.

[9] Collins and Krishna [1995] report similar incentives under an on-campus housing mechanism used at Harvard for the period 1977-1989.

[10] Thomas Tobin, *Yep, it's complicated. If you care where your kid ends up, you have to be savvy and alert.* St. Petersburg Times, September 14, 2003.

4 Nash Equilibria under the Boston Mechanism

In school districts that rely on the Boston mechanism, the students and their parents play a non-trivial preference revelation game. Under this game, the strategies of the students are preferences over schools and the outcome is determined by the Boston mechanism. The choice of their stated preferences and especially their stated top choices play a key role in determining the schools they will be assigned. In our main result we characterize the set of Nash equilibrium outcomes of the preference revelation game induced by the Boston mechanism. Before we present our main result, we give a detailed example which illustrates the preference revelation game induced by the Boston mechanism and highlights some of the key points.

Example 2: There are three students i_1, i_2, i_3 and three schools a, b, c each of which has one seat. The utilities of the students as well as their priorities are as follows:

$$\begin{array}{c} \\ U_{i_1} \\ U_{i_2} \\ U_{i_3} \end{array} \begin{array}{|c|c|c|} \multicolumn{1}{c}{a} & \multicolumn{1}{c}{b} & \multicolumn{1}{c}{c} \\ \hline 2 & 1 & 0 \\ \hline 1 & 2 & 0 \\ \hline 0 & 2 & 1 \\ \hline \end{array} \qquad \begin{array}{l} f_a : i_3 - i_2 - i_1 \\ f_b : i_1 - i_3 - i_2 \\ f_c : i_2 - i_1 - i_3 \end{array}$$

Each student can submit one of the preferences abc, acb, bac, bca, cab, cba and therefore under the Boston mechanism the following $6 \times 6 \times 6$ simultaneous game is induced:

[Figure 1 about here]

In the resulting game the payoff vectors $(2, 2, 1)$, $(2, 0, 2)$, $(1, 1, 1)$, $(1, 0, 0)$, $(0, 1, 2)$, and $(0, 2, 0)$ correspond to the matchings

$$\mu_1 = \begin{pmatrix} i_1 & i_2 & i_3 \\ a & b & c \end{pmatrix}, \ \mu_2 = \begin{pmatrix} i_1 & i_2 & i_3 \\ a & c & b \end{pmatrix}, \ \mu_3 = \begin{pmatrix} i_1 & i_2 & i_3 \\ b & a & c \end{pmatrix},$$

$$\mu_4 = \begin{pmatrix} i_1 & i_2 & i_3 \\ b & c & a \end{pmatrix}, \ \mu_5 = \begin{pmatrix} i_1 & i_2 & i_3 \\ c & a & b \end{pmatrix}, \ \text{and} \ \mu_6 = \begin{pmatrix} i_1 & i_2 & i_3 \\ c & b & a \end{pmatrix} \text{ respectively.}$$

In the resulting game the boldface payoff vectors correspond to Nash equilibria. We have two key observations about the Nash equilibria:

1. The strategy profile which corresponds to truthful preference revelation, (abc, bac, bca), is NOT a Nash equilibrium of the induced preference revelation game.

2. The payoff vector in Nash equilibria is either $(1, 1, 1)$ which is the payoff for matching μ_3, or $(1, 0, 0)$ which is the payoff for matching μ_4. The significance of matchings μ_3 and μ_4 is that they constitute the set of stable matchings under true preferences.[11]

[11]The matching μ_1 is blocked by the student-school pair (i_3, b), the matching μ_2 is blocked by the student-school pair (i_2, a), the matching μ_5 is blocked by the student-school pair (i_1, b), and the matching μ_6 is blocked by the student-school pair (i_1, b).

The matching μ_3 is the student optimal stable matching for the true preferences. Note that the unstable matching μ_1 Pareto dominates the student optimal stable matching μ_3 which in turn Pareto dominates the other stable matching μ_4. The reason that neither of the stable matchings is Pareto efficient is because stability and Pareto efficiency are not compatible in the context of school choice.[12] That is, the efficiency loss going from μ_1 to μ_3 is due to this incompatibility, therefore it can not be recovered given that we are required to respect the legally determined priorities. However the efficiency loss going from μ_3 to μ_4 would be caused by student families being stuck in a bad equilibrium. This additional efficiency loss *can be recovered* by employing the student optimal stable mechanism instead of the Boston mechanism.

We are now ready to present our main result which shows that these observations are not specific to the above example. The key to this result is the similarity between participation of a student in blocking of an unstable matching and profitable deviation by a student in the game induced by the Boston mechanism.

Theorem 1 *Let P_I be the list of true student preferences and consider the preference revelation game induced by the Boston mechanism. The set of Nash equilibrium outcomes of this game is equal to the set of stable matchings under the true preferences P_I.*

Proof: Let $Q = (Q_1, \ldots, Q_n)$ be an arbitrary strategy profile and let μ be the resulting outcome of the Boston mechanism. Suppose μ is not stable under the true preferences. Then there is a student-school pair (i, s) such that student i prefers school s to his assignment $\mu(i)$ and either school s has an empty seat under μ, or student i has higher priority at school s than another student who is assigned a seat at school s. This implies that under the stated preference Q_i student i does not rank school s as his first choice for otherwise he would be assigned a seat at school s. Let Q'_i be any strategy where student i ranks school s as his first choice. Student i is assigned a seat at school s under the profile (Q_{-i}, Q'_i) and therefore neither Q is a Nash equilibrium profile nor μ is a Nash equilibrium outcome. Hence any Nash equilibrium outcome should be stable under the true preferences.

Conversely let μ be a stable matching under the true preferences. Consider a preference profile $Q = (Q_1, \ldots, Q_n)$ where each student i ranks school $\mu(i)$ as his top choice under his stated preferences Q_i. Under the preference profile Q, the Boston mechanism terminates at Round 1 and each student is assigned a seat at his first choice based on the stated preferences. Hence μ is the resulting outcome for the strategy profile Q. Next we show that Q is a Nash equilibrium profile. Consider a student i and a school s such that student i prefers school s to his assignment $\mu(i)$. Since μ is stable, not only all seats of school s are filled under μ but also each student who

[12] Ergin [2002] identifies a condition on the list of priority orderings that is both necessary and sufficient for the compatibility of Pareto efficiency and stability. In contrast, any stable matching is Pareto efficient in the context of two-sided matching markets where the welfare of both sides of the market are taken into account.

is assigned a seat at school s under μ has higher priority than student i for school s. Moreover each such student j ranks school s as his first choice under Q_j. Therefore given Q_{-i}, there is no way student i can secure a seat at school s even if he ranks it as his first choice. Therefore Q is a Nash equilibrium strategy profile and μ is a Nash equilibrium outcome. Hence any stable matching under the true preferences is a Nash equilibrium outcome. $\diamondsuit$

Even though the Boston mechanism itself is not stable, by Theorem 1 all Nash equilibrium outcomes of the preference revelation game induced by the Boston mechanism are stable. The outcome of the Boston mechanism is Pareto efficient, provided that students truthfully reveal their preferences. However truthful preference revelation is rarely in the best interest of students, and efficiency loss is expected. Theorem 1 clarifies the nature of this efficiency loss: Since all equilibrium outcomes are stable, part of the inefficiency is due to the incompatibility of efficiency and stability. However out of all equilibrium outcomes there is one, the student-optimal stable matching, which Pareto dominates any other. Therefore in all equilibrium outcomes with the exception of the student-optimal stable matching there is additional efficiency loss.

Researchers in education tend to evaluate the Boston mechanism and its variants based on the stated preferences of students. For example Glenn [1991] argues that in 1991, 74 percent of sixth graders at Boston were assigned to their first choice school. He also states

> As an example of how school selections change, analysis of first-place preferences in Boston for sixth-grade enrollment in 1989 (the first year of controlled choice in Boston) and 1990 shows that the number of relatively popular schools doubled in only the second year of controlled choice. The strong lead of few schools was reduced as others "tried harder."

Given the incentives under the Boston mechanism, this conclusion is overly optimistic. A more plausible scenario is, the first time the mechanism was implemented most families did not understand the details of the mechanism and naively revealed their preferences truthfully; by the second year of implementation the incentives offered by the mechanism was understood and most families stated their preferences strategically. Along similar lines Glazerman and Meyer [1994] argue that in 1993-94 more than 80 percent of students at Minneapolis were assigned to their first choice school and they conclude

> These numbers imply that student preferences in Minneapolis are quite diverse and that students perceive that there are significant differences in school characteristics. If this were not the case, we might expect most students to apply to a very limited set of schools. As a result, very few students would have been assigned to their preferred school.

Once again, this conclusion is inadequate. The Boston mechanism gives each student an incentive to state a preference in which he top ranks the best possible school that he can be assigned to, given the submitted preferences of other students. However this best possible school is not necessarily the true top choice. It is interesting to note that, under the Nash equilibrium strategies we constructed, each students is assigned to his/her first choice school based on the stated preferences. Whether intentional or not, school districts that use the Boston mechanism are misleading policy-makers by giving the impression that they are able to accommodate most students' top choices and ironically the Boston mechanism is the perfect tool to create this impression.

5 The Two-Sided Case: Strategic Schools

In some cities such as the New York City, schools determine their priority rankings subject to certain regulatory restrictions. In this case it is natural to expect that schools will behave strategically when submitting their priorities. Since both sides of the market are strategic we will call this the two-sided case. We will show that given the Boston mechanism and under suitable assumptions, it is a dominant strategy for schools to submit their true preferences over students as their priority rankings. As a corollary, our main result extends to the two-sided case when schools play undominated strategies.

Let $P_S = (P_s)_{s\in S}$ denote the list of strict school preferences over subsets of students where each subset corresponds to an incoming class. We assume that for any school s, for any subset of students J and any two students $i, j \notin J$, iP_sj implies $(J \cup i)\ P_s\ (J \cup j)$. This property is known as **responsiveness** (Roth [1985]) and it is a consistency condition between preferences over individual students and over sets of students. Two priority rankings f_s and f'_s of school s are **outcome equivalent** if for any list of student preferences P_I and any list of priority rankings of the remaining schools f_{-s}, the Boston mechanism yields the same matching for (P_I, f_{-s}, f_s) and (P_I, f_{-s}, f'_s).

Theorem 2 *In the two-sided version of the Boston mechanism, it is a dominant strategy for any school s to rank students based on its true preferences P_s. Moreover any other dominant strategy of school s is outcome equivalent to truthfully ranking students based on P_s.*

Proof: Let f^*_s rank students based on P_s. Let Q_I be a list of student preferences and f_{-s} be a list of school priorities for all schools but school s. Let f_s be an arbitrary priority order and consider the outcome of the Boston mechanism for (Q_I, f_{-s}, f_s). If school s does not fill its capacity under the resulting matching then the algorithm does not depend on f_s, hence it would yield the same matching for any priority order. If on the other hand s fills its capacity, then let k^* be the round where the last seat in s is assigned. Note that the assignments in rounds earlier than k^* do not depend on f_s. At the beginning of round k^*, let J be the set of students who are already assigned a seat at s, K be the set of unassigned students who rank s as their $k^{*\text{th}}$ choice, r be the number of remaining seats at s, and L be the set of top r individual students in K based on P_s. By responsiveness $(J \cup L)\ R_s\ (J \cup L')$ for any r student subset L' of K and indifference occurs only when $L = L'$. Given Q_I and f_{-s}, if s submits f_s^* its remaining seats are assigned to L, and if s submits any other priority order $f_s \neq f_s^*$ its seats are assigned to L' for some r student subset L' of K. Therefore s is weakly better-off submitting f_s^* than submitting f_s, and indifferent only if $L = L'$, when (Q_I, f_{-s}, f_s) and (Q_I, f_{-s}, f_s^*) yield the same matching.

Since the initial choice of Q_I and f_{-s} was arbitrary, we conclude that it is a dominant strategy for s to submit f_s^* and that any other dominant strategy f_s must be outcome equivalent to f_s^*. $\diamondsuit$

Corollary 1 *In the two-sided version of the Boston mechanism, the set of Nash equilibrium outcomes in undominated strategies is equal to the set of stable matchings under the true preferences.*

Remark 1: Consider a two-sided matching market where each participant has a capacity of one and refer the two sides of the market as men and women. Consider the preference revelation game induced by the man-optimal stable mechanism. The combination of a result by Roth [1984b] and another by Gale and Sotomayor [1985] in this context is analogous to Corollary 1: Under the man-optimal stable mechanism the set of Nash equilibrium outcomes in undominated strategies is equal to the set of stable matchings under the true preferences.

6 Controlled Choice

One of the major concerns about the implementation of school choice programs is that they may result in racial and ethnic segregation at schools. Because of these concerns, school choice programs in some districts are limited by court-ordered desegregation guidelines. This version of school choice is known as **controlled choice**. In Minneapolis, controlled choice constraints are implemented by imposing type-specific quotas. Under this practice students are partitioned into different groups based on their type (which often depends on their ethnic/racial background) and for each school, type-specific quotas are determined in additional to the capacity of the school. These quotas may be rigid or they may be flexible. For example, in Minneapolis the district is allowed to go above or below the district-wide average enrollment rates by up to 15 percent points in determining the ethnic/racial quotas. Currently in Minneapolis and for ten years until 1999 in Boston the following variant of the Boston mechanism is used to assign students to public schools.

The Boston Mechanism with Type-Specific Quotas: The students are partitioned based on their types and for each school, in addition to the capacity of the school, type-specific quotas are determined. For each school a strict priority ordering of the students is determined, each student submits a preference ranking of the schools, and based on type-specific quotas, student priorities, and submitted preferences, the student assignment is determined in several rounds.

Round 1: In Round 1 only the first choices of the students are considered. For each school, consider the students who have listed it as their first choice and assign seats of the school to these students one at a time following their priority order unless the quota of a type is full. When that happens, remaining students of that type are rejected and the process continues with the students of other types until either there are no seats left or there is no student left who has listed it as his first choice.

In general, at

Round k: Consider the remaining students. In Round k only the k^{th} choices of these students are considered. For each school with still available seats, consider the students who have listed it as their k^{th} choice and assign the remaining seats to these students one at a time following their priority order unless the quota of a type is full. When that happens, remaining students of that type are rejected and the process continues with the students of other types until either there are no seats left or there is no student left who has listed it as his k^{th} choice.

The procedure terminates when each student is assigned a seat at a school.

As in the case of the Boston mechanism, this modified version also induces a non-trivial preference revelation game. We need an additional definition in order to characterize the set of Nash equilibrium outcomes of this game.

Given a controlled choice problem, we call a matching μ **weakly stable** if it does not violate the type-specific quotas, and there is no student-school pair (i, s) such that student i prefers school s to his assignment $\mu(i)$ and either (a) school s has not filled its quota for the type of student i and it has an empty seat, or (b) school s has not filled the quota for the type of student i and student i has higher priority than another student (of any type) who is assigned a seat at school s, or (c) school s has filled its quota for the type of student i but student i has higher priority than another student of his own type who is assigned a seat at school s. Following Kelso and Crawford [1982] and Roth [1991], Abdulkadiroğlu [2002] shows that the set of weakly stable matchings is non-empty. We are ready to characterize the set of Nash equilibrium outcomes of the preference revelation game induced by the Boston mechanism with type-specific quotas.

Theorem 3 *Let P_I be the list of true student preferences and consider the preference revelation game induced by the Boston mechanism with type-specific quotas. The set of Nash equilibrium outcomes of this game is equal to the set of weakly stable matchings under the true preferences P_I.*

Proof: Similar to the proof of Theorem 1.

Many of the key properties on the structure of stable matchings carry over to the set of weakly stable matchings provided that types of students form a partition of the students (see Abdulkadiroğlu [2002]). Most notably, given a controlled choice problem, there exists a weakly stable matching which is at least as good as any other weakly stable matching for any student (Kelso and Crawford [1982], Roth [1991], Abdulkadiroğlu [2002]). Theorem 3 shows that policy implications of our main result carry over to the controlled choice model. Most notably, transition to the controlled choice version of the student-optimal stable mechanism is likely to result in Pareto improvements in school districts that currently rely on the Boston mechanism with type-specific quotas.[13]

7 Nash Equilibria Under Priority Matching Mechanisms

As we have already indicated, the Boston mechanism is a special case of priority matching mechanisms. A natural question is whether our characterization result extends to other priority matching mechanisms. The following example shows that the answer is negative. Indeed there is a priority matching mechanism and a school choice problem where the set of stable matchings and the set of Nash equilibrium outcomes of the induced preference revelation game are two distinct sets.

Example 3: Let $I = \{i_1, i_2\}$ be the set of students, $S = \{a, b\}$ be the set of schools, and $q = (2, 2)$ be the school capacity vector. Student priorities at schools and their preferences are as follows:

$$\begin{array}{ll} f_a : i_1 - i_2 & \qquad P_{i_1} : b - a \\ f_b : i_2 - i_1 & \qquad P_{i_2} : a - b \end{array}$$

Note that the unique stable matching for this problem is:

$$\mu_1 = \begin{pmatrix} i_1 & i_2 \\ b & a \end{pmatrix}.$$

Next consider the priority matching mechanism which first considers school priorities and only then the student preferences. This mechanism

- first forms any feasible (1,1) match,
- next forms any feasible (2,1) match,
- next forms any feasible (1,2) match,
- and finally forms any feasible (2,2) match

when there are two students and two schools. Observe that given the above priorities at schools, the outcome of this mechanism is

$$\mu_2 = \begin{pmatrix} i_1 & i_2 \\ a & b \end{pmatrix}$$

regardless of the stated student preferences. Hence any preference profile is a Nash equilibrium with an outcome of μ_2.

[13] Roth [1991] reports that a similar transition had been carried out in Edinburgh hospital-intern market in 1969 where a priority matching mechanism was replaced with the controlled choice version of a stable mechanism.

While the above example is discouraging, a minor modification in the school choice model allows us to generalize our characterization result to priority matching mechanisms. In the original model each student ranks all schools and she does not have the ability to "veto" any school. In practice, however, students often have outside options (such as private schools) and they are allowed to consider any subset of schools. We next modify the school choice model to allow for this possibility.

In this richer model each student i has strict preferences P_i over $S \cup \{i\}$ where i denotes the option of remaining unmatched. Let R_i denote the weak preference relation induced by P_i. School s is **acceptable** to student i if and only if $sR_i i$.

A **matching** in this modified model is a function $\mu\colon I \to S \cup I$ such that

(i) $\mu(i) \in S \cup \{i\}$ for all $i \in I$, and

(ii) $|\mu^{-1}(s)| \leq q_s$ for all $s \in S$

and it is **stable** if and only if

(a) $\mu(i)R_i i$ for any student i,

(b1) there is no student-school pair (i, s) and another student j with $\mu(j) = s$ such that $sP_i\mu(i)$ and $f_s(i) < f_s(j)$, and

(b2) there is no student-school pair (i, s) such that $sP_i\mu(i)$ and $|\mu^{-1}(s)| < q_s$.

A **priority matching mechanism** is defined similarly as in the original model with the exception that students are only admitted to acceptable schools. Recall that a $\boldsymbol{(k, l)}$ **match** is defined to be a match between a student-school pair such that the student ranks the school k^{th} in his preferences and he has the l^{th} priority at the school. Given a modified problem with n students and m schools, a **match priority** is a one-to-one function

$$\pi\colon \{1, \ldots, n\} \times \{1, \ldots, m\} \to \{1, \ldots, nm\}$$

and the resulting priority matching mechanism determines its outcome in nm steps with the following **priority matching algorithm**:

Step 1: Form any feasible and acceptable $\pi^{-1}(1)$ match.

Step 2: Form any feasible and acceptable $\pi^{-1}(2)$ match.

$\vdots \qquad \vdots$

Step nm: Form any feasible and acceptable $\pi^{-1}(nm)$ match.

Each student who remains unmatched at the end of nm steps is matched to herself.

For example under the special case of the Boston mechanism any feasible and acceptable $\pi^{-1}(1)$=(1,1) match is formed at Step 1, any feasible and acceptable $\pi^{-1}(2)$=(1,2) match is formed at Step 2, etc. Note that the first n steps under this description correspond to Round 1 of the original description of the Boston mechanism, the next n steps correspond to Round 2, and so on.

A match priority π is **monotonic** if $(k,l) \leq (k',l')$ implies $\pi(k,l) \leq \pi(k',l')$. A priority matching mechanism is **monotonic** if it is induced by a monotonic match priority.

We are now ready to present our final result.

Theorem 4 *Consider the modified school choice model where each student can consider any subset of the schools. Let P be a list of student preferences and consider the preference revelation game induced by any monotonic priority matching mechanism. The set of Nash equilibrium outcomes of this game is equal to the set of stable matchings under the true preferences P.*

Proof: Fix a modified problem where P denotes the list of student preferences. Fix a monotonic match priority π and let Π denote the resulting priority matching mechanism. Consider the induced preference revelation game.

"$\supset$": First suppose that μ is a stable matching under P. For each student i with $\mu(i) = i$, let $\tilde{P}_i$ be a preference ranking with no acceptable school. For each student i with $\mu(i) = s$, let $\tilde{P}_i$ be a preference ranking where s is the only acceptable school. Clearly $\Pi(\tilde{P}) = \mu$.

We next show that $\tilde{P}$ is a Nash equilibrium. Suppose towards a contradiction that there is a student i, a preference ranking P'_i, and a school s such that $\Pi_i(\tilde{P}_{-i}, P'_i) = s$ and $sP_i\mu(i)$. Since μ is stable, there are q_s students $j_1, \ldots, j_{q_s}$ each of whom is assigned a seat at school s under μ and also has a higher priority for school s than student i. Consider the priority matching algorithm under $(\tilde{P}_{-i}, P'_i)$. School s is the only acceptable school for each student $j \in \{j_1, \ldots, j_{q_s}\}$ under $\tilde{P}_j$ and by monotonicity of the match priority π, the match of each such student j and school s has higher match priority than the match of student i and school s. Therefore the seats of school s are exhausted before the match of i and s is considered, achieving the desired contradiction. Hence $\tilde{P}$ is a Nash equilibrium and μ is a Nash equilibrium outcome.

"$\subset$": Let $\tilde{P}$ be a strategy profile that yields the unstable matching μ under the priority matching mechanism Π. We will show that $\tilde{P}$ is not a Nash equilibrium for each of the following three cases:

(a) there is a student i^* such that $i^* P_{i^*} \mu(i^*)$,

(b1) there is a student-school pair (i^*, s^*) and another student j^* with $\mu(j^*) = s^*$ such that $s^* P_{i^*} \mu(i^*)$ and $f_{s^*}(i^*) < f_{s^*}(j^*)$, and

(b2) there is no student-school pair (i^*, s^*) such that $s^* P_{i^*} \mu(i^*)$ and $|\mu^{-1}(s^*)| < q_{s^*}$.

(a) Let P'_{i^*} be a preference ranking where no school is acceptable. The priority matching mechanism Π leaves student i unmatched under the profile $(\tilde{P}_{-i^*}, P'_{i^*})$ and hence P'_{i^*} is a profitable deviation.

(b) Let P'_{i^*} be a preference ranking where the only acceptable school is s^*. Let l denote the priority ranking of student i^* at school s^* (i.e. $l = f_{s^*}(i^*)$) and let r^* denote the step at which all feasible and acceptable (1,l) matches are formed by the priority matching algorithm for the match priority π (i.e. $r^* := \pi(1,l)$). We will show by induction that:

Claim. Consider the priority matching algorithm for the match priority π. At the beginning of each round r ($1 \leq r \leq r^*$):

1. For each student $i \neq i^*$, if i is already matched under $\tilde{P}$, then he is also already matched under $(\tilde{P}_{-i^*}, P'_{i^*})$.

2. For each school s, there are at least as many unassigned seats under $(\tilde{P}_{-i^*}, P'_{i^*})$ as under $\tilde{P}$.

Proof of the Claim: Since the priority matching algorithm starts with each student unmatched, the Claim holds for $r = 1$. Suppose the Claim holds for r where $1 \leq r < r^*$. We will show that it holds for $(r+1)$ as well.

1. Take any student $i \neq i^*$ who gets matched to a school, say school s, at Step r under $\tilde{P}$. We will show that student i is matched to a school by the end of Step r under $(\tilde{P}_{-i^*}, P'_{i^*})$. School s has at least one available seat at the beginning of Step r under $\tilde{P}$ and therefore by part 2 of the inductive assumption it has at least one available seat at the beginning of Step r under $(\tilde{P}_{-i^*}, P'_{i^*})$ as well. Suppose student i is still unmatched at the beginning of Step r under $(\tilde{P}_{-i^*}, P'_{i^*})$. Since $i \neq i^*$, student i and school s form a $\pi^{-1}(r)$ match under $(\tilde{P}_{-i^*}, P'_{i^*})$ and hence student i gets matched by the end of Step r under $(\tilde{P}_{-i^*}, P'_{i^*})$.

2. Take any school s and consider any student i who is matched with school s at Step r under $(\tilde{P}_{-i^*}, P'_{i^*})$. We will show that either student i is matched with school s at Step r under $\tilde{P}$ or there are no seats left at school s at the beginning of Step r under $\tilde{P}$. Recall that under P'_{i^*} the only acceptable school is s^*, and by assumption i^* and s^* can only be matched at Step r^* (in case a seat is still available at s^*). Therefore $i \neq i^*$. By assumption student i is unmatched at the beginning of Step r under $(\tilde{P}_{-i^*}, P'_{i^*})$ and therefore by part 1 of the inductive assumption student i is unmatched at the beginning of Step r under $\tilde{P}$ as well. Hence if s has any seats left at the beginning of Step r under $\tilde{P}$, then student i and school s form a $\pi^{-1}(r)$ match and get matched at Step r.

This completes the proof of the claim. $\diamond$

Recall that by assumption student i^* is unmatched at the beginning of Step r^* under $(\tilde{P}_{-i^*}, P'_{i^*})$. Suppose (b1) holds and student j^* and school s^* form a (k, l') match under $\tilde{P}$ for some $l < l'$. By monotonicity of the priority matching mechanism Π a $(1, l)$ match will be considered before a (k, l') match for any k, and hence school s^* has an empty seat at the beginning of Step r^* (i.e. when $(1, l)$ matches are considered) under $\tilde{P}$. If on the other hand (b2) holds, then again school s^* has an empty seat at the beginning of Step r^* (and indeed throughout the algorithm) under $\tilde{P}$. Therefore by the above Claim, school s^* has an empty seat at the beginning of Step r^* under $(\tilde{P}_{-i^*}, P'_{i^*})$ whether (b1) or (b2) holds and student i^* and school s^* are matched at Step r^* under $(\tilde{P}_{-i^*}, P'_{i^*})$. Hence P'_{i^*} is a profitable deviation for student i^* showing that $\tilde{P}$ is not a Nash equilibrium. $\diamondsuit$

8 Incomplete Information

So far we relied on Nash equilibrium (and Nash equilibrium in undominated strategies) in our equilibrium analysis and hence we assumed complete information about the preferences. As we have shown our result is quite robust for complete information environments and a natural question is what happens if the complete information assumption is relaxed. We next show that our characterization does not carry over to an incomplete information environment and moreover a student may be better off under the Boston mechanism than under the student-optimal stable mechanism.

Example 4: Let $I = \{i_1, i_2, i_3\}$ be the set of students, $S = \{a, b, c\}$ be the set of schools, and $q = (1, 1, 1)$ be the school capacity vector. Suppose that all three schools have the same priority ranking

$$f_a = f_b = f_c : i_1 - i_2 - i_3.$$

All students are expected utility maximizers and while the **types** (i.e. utility functions) U_{i_2}, U_{i_3} of students i_2, i_3 are known with certainty, student i_1 is of one of the three types $U^a_{i_1}$, $U^b_{i_1}$, $U^c_{i_1}$ with probabilities 1/4, 1/4, 1/2 respectively. The student types are as follows:

	$U^a_{i_1}$	$U^b_{i_1}$	$U^c_{i_1}$
a	2	1	1
b	1	2	0
c	0	0	2

	U_{i_2}
a	2
b	1
c	0

	U_{i_3}
a	1
b	2
c	0

Consider the preference revelation game induced by the Boston mechanism and observe that truth-telling, i.e. the strategy where

- student i_1 reports $a - b - c$ when he is of type $U^a_{i_1}$, $b - a - c$ when he is of type $U^b_{i_1}$, and $c - a - b$ when he is of type $U^c_{i_1}$,
- student i_2 reports $a - b - c$, and
- student i_3 reports $b - a - c$

is a Bayesian Nash equilibrium with an expected payoff vector of $(2, \frac{3}{2}, \frac{3}{2})$.[14] The outcome of this equilibrium is a lottery but not all matchings in its support are stable. In particular, when the realized type profile is $(U^a_{i_1}, U_{i_2}, U_{i_3})$, truth-telling yields

$$\begin{pmatrix} i_1 & i_2 & i_3 \\ a & c & b \end{pmatrix}$$

which is an unstable matching.

The following table compares the expected payoffs of the dominant-strategy equilibrium of the student-optimal stable mechanism and the above described Bayesian Nash equilibrium of the Boston mechanism:

	$U^a_{i_1}$	$U^b_{i_1}$	$U^c_{i_1}$	U_{i_2}	U_{i_3}
Student-Optimal Stable Mechanism	2	2	2	$\frac{7}{4}$	1
The Boston Mechanism	2	2	2	$\frac{3}{2}$	$\frac{3}{2}$

Note that the Bayesian Nash equilibrium described above benefits the low-priority student i_3 at the expense of the intermediate priority student i_2.

9 Conclusion

In this paper we presented an equilibrium analysis of the Boston mechanism, an assignment mechanism that is in use at several U.S. school districts including Boston, Cambridge, Charlotte, Minnesota, Seattle, and St. Petersburg-Tampa. Our results suggest that a transition to an alternative mechanism, the student-optimal stable mechanism, is likely to result in potentially significant welfare gains. Such a transition will also eliminate the need for strategizing because truthful preference revelation is a dominant strategy under the student-optimal stable mechanism. In contrast, as we present, the Boston mechanism induces a complicated coordination game with multiple equilibria among large numbers of parents. Unlike in complete information environments, some students may benefit from the Boston mechanism in incomplete information environments due to coordination failures of other students. As it is recently argued by the Boston Public Schools Strategic Planning Manager,[15] "assignment becomes a high-stakes gamble for families"

[14] Since student i_1 has the highest priority at each school, truthful preference revelation yields him his top choice regardless of his type. It is also clear that no student can profit from improving the ranking of his last choice school. That leaves $b-a-c$ as the only potentially profitable deviation for student i_2 and $a-b-c$ as the only potentially profitable deviation for student i_3. However using strategy $b-a-c$ reduces the expected utility of student i_2 to $\frac{5}{4}$ and using strategy $a-b-c$ reduces the expected utility of student i_3 to 1 showing that truth-telling is a Bayesian Nash equilibrium.

[15] Valerie Edwards, "Understanding the Options for a New BPS Assignment Method," presented at the October 13, 2004 dated Boston Public Schools school committee meeting.

under the Boston mechanism. One important direction for future research is a thorough analysis of equilibria in incomplete information environments for it will enhance our understanding of this high stakes gamble.

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abc

	abc	*acb*	*bac*	*bca*	*cab*	*cba*
abc	1,0,0	1,0,0	0,2,0	0,2,0	1,0,0	1,0,0
acb	0,2,0	0,2,0	0,2,0	0,2,0	1,0,0	1,0,0
bac	1,0,0	1,0,0	1,0,0	1,0,0	**1,0,0**	**1,0,0**
bca	1,0,0	1,0,0	1,0,0	1,0,0	**1,0,0**	**1,0,0**
cab	0,2,0	0,2,0	0,2,0	0,2,0	1,0,0	1,0,0
cba	0,2,0	0,2,0	0,2,0	0,2,0	1,0,0	1,0,0

acb

	abc	*acb*	*bac*	*bca*	*cab*	*cba*
abc	1,0,0	1,0,0	0,2,0	0,2,0	1,0,0	1,0,0
acb	0,2,0	0,2,0	0,2,0	0,2,0	1,0,0	1,0,0
bac	1,0,0	1,0,0	1,0,0	1,0,0	**1,0,0**	**1,0,0**
bca	1,0,0	1,0,0	1,0,0	1,0,0	**1,0,0**	**1,0,0**
cab	0,2,0	0,2,0	0,2,0	0,2,0	1,0,0	1,0,0
cba	0,2,0	0,2,0	0,2,0	0,2,0	1,0,0	1,0,0

bac

	abc	*acb*	*bac*	*bca*	*cab*	*cba*
abc	0,1,2	0,1,2	2,0,2	2,0,2	2,0,2	2,0,2
acb	0,1,2	0,1,2	2,0,2	2,0,2	2,0,2	2,0,2
bac	**1,1,1**	**1,1,1**	1,0,0	1,0,0	1,0,0	1,0,0
bca	**1,1,1**	**1,1,1**	1,0,0	1,0,0	1,0,0	1,0,0
cab	0,1,2	0,1,2	0,1,2	0,1,2	2,0,2	2,0,2
cba	0,1,2	0,1,2	0,1,2	0,1,2	2,0,2	2,0,2

bca

	abc	*acb*	*bac*	*bca*	*cab*	*cba*
abc	0,1,2	0,1,2	2,0,2	2,0,2	2,0,2	2,0,2
acb	0,1,2	0,1,2	2,0,2	2,0,2	2,0,2	2,0,2
bac	**1,1,1**	**1,1,1**	1,1,1	1,0,0	1,0,0	1,0,0
bca	**1,1,1**	**1,1,1**	1,1,1	1,0,0	1,0,0	1,0,0
cab	0,1,2	0,1,2	0,1,2	0,1,2	2,0,2	2,0,2
cba	0,1,2	0,1,2	0,1,2	0,1,2	2,0,2	2,0,2

cab

	abc	*acb*	*bac*	*bca*	*cab*	*cba*
abc	1,1,1	1,1,1	2,2,1	2,2,1	2,0,2	2,0,2
acb	1,1,1	1,1,1	2,2,1	2,2,1	2,0,2	2,0,2
bac	**1,1,1**	**1,1,1**	1,1,1	1,1,1	1,0,0	1,0,0
bca	**1,1,1**	**1,1,1**	1,1,1	1,1,1	1,0,0	1,0,0
cab	0,1,2	0,1,2	0,1,2	0,1,2	1,0,0	1,0,0
cba	0,1,2	0,1,2	0,1,2	0,1,2	1,0,0	1,0,0

cba

	abc	*acb*	*bac*	*bca*	*cab*	*cba*
abc	1,1,1	1,1,1	2,2,1	2,2,1	2,0,2	2,0,2
acb	1,1,1	1,1,1	2,2,1	2,2,1	2,0,2	2,0,2
bac	**1,1,1**	**1,1,1**	1,1,1	1,1,1	1,0,0	1,0,0
bca	**1,1,1**	**1,1,1**	1,1,1	1,1,1	1,0,0	1,0,0
cab	0,1,2	0,1,2	0,1,2	0,1,2	2,0,2	2,0,2
cba	0,1,2	0,1,2	0,1,2	0,1,2	1,0,0	1,0,0

Figure 1: The simultaneous game induced by the Boston mechanism for the school choice problem in Example 1. In this game i_1 is the row player, i_2 is the column player and i_3 is the matrix player.