Icosahedral Polynomials

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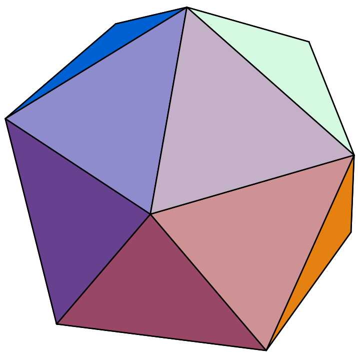

ICOSAHEDRAL POLYNOMIALS

Abstract. A polynomial is said to be invariant for a group of linear fractional transformations G if its roots are permuted by G. We begin by using a simple group of linear fractional transformations that is isomorphic to S_3 and finding its invariant polynomials to build up the tools necessary to attack a larger group. We then follow a construction from Toth [2] of the icosahedral group I, and derive a general formula for all polynomials of degree 60 that are invariant under I.

1. Introduction

Definition 1. *Let $f(z)$ be a polynomial of degree n. $f(z)$ is **palendromic** if*

$$z^n f(1/z) = f(z).$$

Note that if z_0 is a root of $f(z)$ and $z_0 \neq 0$, $1/z_0$ is also a root of $f(z)$. Furthermore, $\bar{z}_0$ and $1/\bar{z}_0$ are also roots of $f(z)$.

Definition 2. *α is an **algebraic number** if $f(\alpha) = 0$ for some $f(z) \in \mathbb{Q}[z]$.*

Let $\bar{\mathbb{Q}} = \{\alpha \mid \alpha$ is an algebraic number of a field in $\mathbb{C}\}$.

Let $\mathbb{G}$ be the group of all field automorphisms of $\bar{\mathbb{Q}}$. For a field F, a field automorphism is a one-to-one and onto mapping $\sigma : F \to F$ that preserves the algebraic properties of F.

Proposition 1. *If $\sigma \in \mathbb{G}$ then*

- $\sigma(1) = 1$
- $\sigma(n) = n \quad \forall\, n \in \mathbb{Z}$
- $\sigma(\frac{a}{b}) = \frac{a}{b} \quad \forall a, b \in \mathbb{Z}, b \neq 0$
- $\mathbb{G}$ *fixes* $\mathbb{Q}$.

Proof. Let F be a field, $a \in F$, and $\sigma \in \mathbb{G}$. Then $\sigma(a) = \sigma(1a) = \sigma(1)\sigma(a)$, so $\sigma(1) = 1$. Let $a \in \mathbb{Z}$. Since $a = 1 + 1 + \ldots + 1$,

$$\sigma(a) = \sigma(1 + 1 + \ldots + 1) = \sigma(1) + \sigma(1) + \ldots + \sigma(1) = 1 + 1 + \ldots + 1 = a,$$

so $\sigma(a) = a$ for $a \in (Z)$. As F is a field, there are multiplicative inverses in F. Thus, for $a \in (Z), a \neq 0$, there is a distinct $1/a \in F$ such that $(a)(1/a) = 1$. It follows that

$$1 = \sigma(1) = \sigma((a)(1/a)) = \sigma(a)\sigma(1/a) = a\sigma(1/a).$$

Thus $\sigma(1/a) = 1/a$. Therefore, for $a, b \in \mathbb{Z}, b \neq 0, \sigma(a/b) = a/b$. Since σ is an arbitrary element of $\mathbb{G}$, and $\sigma(a/b) = a/b$, we see that $\mathbb{G}$ fixes $\mathbb{Q}$. □

In general, any $\sigma \in \mathbb{G}$ fixes the smallest ring that contains 1, which is $\mathbb{Z}$ in this case. Here, however, any such $\sigma \in \mathbb{G}$ also fixes $\mathbb{Q}$. Therefore, for $r \in \mathbb{Q}, \alpha \in \bar{\mathbb{Q}}$,

$$\sigma(r\alpha) = \sigma(r)\sigma(\alpha) = r\sigma(\alpha),$$

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so $\mathbb{G}$ is a collection of $\mathbb{Q}$-linear maps.

If $f(z) \in \mathbb{Q}[z]$, then $f(z) = a_n z^n + a_{n-1} z^{n-1} + \ldots + a_1 z + a_0$ where $a_i \in \mathbb{Q}, i = 0, 1, 2, \ldots, n$. If we let $F(z_0) = 0$ for some $z_0 \in \bar{\mathbb{Q}}$ and $\sigma \in \mathbb{G}$,

$$0 = a_n z_0^n + a_{n-1} z_0^{n-1} + \ldots + a_1 z_0 + a_0.$$

So by applying σ,

$$\sigma(0) = 0 = a_n \sigma(z_0)^n + a_{n-1} \sigma(z_0)^{n-1} + \ldots + a_1 \sigma(z_0) + a_0.$$

Thus $\sigma(z_0)$ is also a root of f, and σ is a permutation of the roots of f. Therefore there is a homomorphism from $\mathbb{G} \to S_n$, where S_n is the symmetric group on n elements.

2. Linear Fractional Transformations and S_3

Definition 3. *Let $GL_2(\mathbb{Q})$ be the set of invertible 2×2 matrices with rational coefficients.*

Definition 4. *Let $\boldsymbol{M}_{\mathbb{Q}}$ be the set of rational* ***Möbius transformations*** *of the extended complex plane $\hat{\mathbb{C}}$, defined as*

$$\gamma(z) = \frac{az + b}{cz + d}, z \in \mathbb{C}, a, b, c, d \in \mathbb{Q}$$

[5].

Proposition 2. *$\boldsymbol{M}_{\mathbb{Q}}$ is a group under composition.*

Proof. Let $\alpha, \beta \in \mathbf{M}_{\mathbb{Q}}, \alpha = \frac{az+b}{cz+d}, \beta = \frac{ez+f}{gz+h}, a, b, c, d, e, f, g, h \in \mathbb{Q}$.

$$\alpha\beta = \frac{a\frac{ez+f}{gz+h} + b}{c\frac{ez+f}{gz+h} + d} = \frac{\frac{aez+af+bgz+bh}{gz+h}}{\frac{cez+cf+dgz+dh}{gz+h}} = \frac{(ae + bg)z + af + bh}{(ce + dg) + cf + dh}.$$

As $ae+bg, af+bf, ce+dg, cf+dh \in \mathbb{Q}$, we see that $\mathbf{M}_{\mathbb{Q}}$ is closed under composition.

Clearly $\frac{1z+0}{0z+1} = z$ is the identity under composition, and we can easily generate an inverse. Using the calculation above, and assuming that a, b, c, d have specific values, solving for an inverse is the equivalent of solving two pairs of linear equations with two variables in each pair. This guarantees that there will be a well defined inverse for each element of $\mathbf{M}_{\mathbb{Q}}$. Associativity follows from the associativity of the standard multiplication on $\mathbb{Q}$ used in the act of composition. Thus $\mathbf{M}_{\mathbb{Q}}$ is indeed a group.

□

Consider $f : GL_2(\mathbb{Q}) \to M_{\mathbb{Q}}$ defined such that $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \to \gamma(z) = \frac{az+b}{cz+d}$ where $ad - bc \neq 0$.

Lemma 1. *f is a homomorphism from $GL_2(\mathbb{Q})$ to $M_{\mathbb{Q}}$*

Proof.

$$f\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} e & f \\ g & h \end{bmatrix}\right) = f\left(\begin{bmatrix} a+e & b+f \\ c+g & d+h \end{bmatrix}\right) = \frac{(ae + bg)z + af + bh}{(ce + dg)z + cf + dh},$$

$$f\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) f\left(\begin{bmatrix} e & f \\ g & h \end{bmatrix}\right) = \frac{a\frac{ez+f}{gz+h} + b}{c\frac{ez+f}{gz+h} + d} = \frac{\frac{aez+af+bgz+bh}{gz+h}}{\frac{cez+cf+dgz+dh}{gz+h}} = \frac{(ae + bg)z + af + bh}{(ce + dg) + cf + dh}.$$

□

Lemma 2. *The kernel of f is αI, where α is a nonzero scalar.*

Proof. First we find the identity element of $M_{\mathbb{Q}}$. Let $e_{M_{\mathbb{Q}}} = \frac{fz+g}{hz+j}$.

$$\frac{az+b}{cz+d} = \frac{az+b}{cz+d} e_{M_{\mathbb{Q}}} = \frac{(af+bh)z + ag + bj}{(cf+dh) + cg + dj} \Rightarrow f = j, f = 0, g = 0, j = f.$$

So now we know that $e_{M_{\mathbb{Q}}} = \frac{\alpha z}{\alpha}$ for $\alpha \in \mathbb{Q}$, $\alpha \neq 0$. Thus the kernel of f is αI for $\alpha \neq 0$. □

Corollary 1. *$GL_2(\mathbb{Q})/\{\alpha I : \alpha$ nonzero in $\mathbb{Q}\}$ is isomorphic to $M_{\mathbb{Q}}$.*

Proof. The homomorphism f is onto by its definition, as an element of $M_{\mathbb{Q}}$ cannot exist without a corresponding matrix in $GL_2(\mathbb{Q})$. It is clear that for $x \in GL_2(\mathbb{Q})$, αx and x, α a nonzero scalar, are sent to the same element of $M_{\mathbb{Q}}$, so an element of $M_{\mathbb{Q}}$ determines the corresponding element of $GL_2(\mathbb{Q})$ up to a scalar. Thus $f : GL_2(\mathbb{Q})/\{\alpha I : \alpha \text{ nonzero in } \mathbb{Q}\} \to M_{\mathbb{Q}}$ is an isomorphism. □

It is interesting to consider subgroups of $M_{\mathbb{Q}}$, and look at how they act on $\hat{\mathbb{C}}$. In particular, a finite subgroup about which we already know a great deal is preferred. Consider the subgroup generated by $\gamma_2 = 1/z$ and $\gamma_3 = 1 - z$. These are two elements of $M_{\mathbb{Q}}$, which are both of order two. We will let $\gamma_1 = \gamma_2^2 = \gamma_3^2 = z$, the identity map. We now generate the rest of the subgroup. $\gamma_4 = \gamma_2\gamma_3 = \frac{1}{1-z}$, $\gamma_5 = \gamma_3\gamma_2 = 1 - 1/z$, and $\gamma_6 = \gamma_3\gamma_2\gamma_3 = \gamma_2\gamma_3\gamma_2 = \frac{z}{z-1}$.

These are all the possible elements that can be generated from the two non-trivial elements and the identity. Thus we have the subgroup

$$\Gamma = \{z, 1/z, 1-z, \tfrac{1}{1-z}, 1 - \tfrac{1}{z}, \tfrac{z}{z-1}\} < M_{\mathbb{Q}}.$$

Lemma 3. *Γ is isomorphic to S_3.*

Proof. We know that Γ is a group with 6 elements. There are only two groups with 6 elements, the cyclic group, which is abelian, and the non-abelian group S_3. As Γ is non-abelian it must be isomorphic to S_3. □

3. How Γ acts on $\hat{\mathbb{C}}$

Consider three circles in $\hat{\mathbb{C}}$: $C_1 = \{z \mid |z| = 1\}, C_2 = \{z \mid |z-1| = 1\}, C_3 = \{z \mid z = 1/2 + bi, b \in \mathbb{R}\}$. The first two circles are clearly circles in the traditional Euclidian sense, but the third is a straight line. However, we can think of this straight line as a circle of infinite radius, or a circle that contains the point at ∞ in the extended complex plane. Since we are considering three circles, S_3 is the group of permutations of the three circles. How do the elements of Γ act on the three circles?

Lemma 4. *Γ permutes C_1, C_2 and C_3.*

Proof. Clearly, γ_1 is the identity and preserves all three circles. $\gamma_2 = 1/z$ fixes C_1 as $1/z = \bar{z}/|z|$. Simple calculations show that γ_2 switches C_2 and C_3. Similarly, γ_6 fixes C_2 while switching C_1 and C_3, and γ_3 fixes C_3 while switching C_1 and C_2. The two remaining γ cycle all three of the elements, with γ_4 sending C_1 to C_3 to C_2 back to C_1 and γ_5 sending C_1 to C_2 to C_3 back to C_1. □

Notice that C_1, C_2 and C_3 divide the complex plane into six disjoint sections.

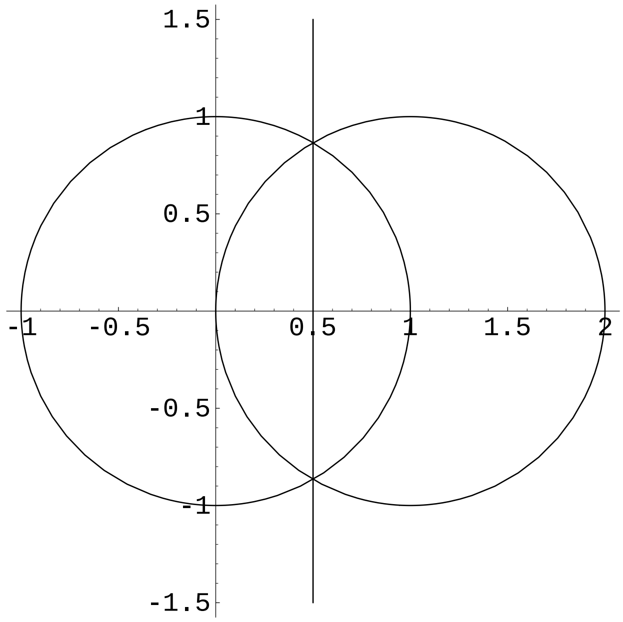

FIGURE 1. $\mathbb{C}_1$, $\mathbb{C}_2$ and $\mathbb{C}_3$.

Lemma 5. *The complex plane is composed of S_3 translations of a fundamental domain.*

Proof. Since the elements of Γ permute the three circles, if x is in one section of the plane and $\gamma(x)$ is in another for $\gamma \in \Gamma$, then the entire first section is mapped to the second section by γ. So it will suffice to pick a point that is not on one of the circles, and follow it through the various operations in Γ. We will look at the point $x = -10 + 0i$.

- $\gamma_1(-10) = -10$
- $\gamma_2(-10) = -1/10$
- $\gamma_3(-10) = 11$
- $\gamma_4(-10) = 1/11$
- $\gamma_5(-10) = 11/10$
- $\gamma_6(-10) = 10/11$.

It is easy to check that these six values are in the six different sections of the plane as divided by C_1, C_2 and C_3. Thus the elements of Γ translate the fundamental domain to each of the six sections created by C_1, C_2 and C_3.

□

Definition 5. *If $f(x) \in \mathbb{Q}[z]$ and $deg(f) \leq k$, and*

$$(cz+d)^k f\left(\frac{az+b}{cz+d}\right) = f(z), \qquad \forall\ \frac{az+b}{cz+d} \in \Gamma,$$

*we say that f is Γ-**modular of weight** k.*

If a polynomial f is S_3-modular, then the roots of f are permuted by Γ.

Lemma 6. *If $f(z)$ is a degree 6 S_3-modular polynomial with complex roots, then the roots must live on C_1, C_2 and C_3.*

Proof. Let $f(z)$ be a degree 6 S_3-modular polynomial with a complex root z_0. We know that the elements of Γ permute the roots of $f(z)$, and we also know that since $f(z) \in \mathbb{Q}[z]$, $\bar{z}_0$ is also a root of f. Complex conjugation is an action of degree two, and one of the elements of Γ must send z_0 to $\bar{z}_0$. As this element conjugates the root, it must also have order two. There are three such elements, $\frac{1}{z}, 1-z, \text{and} \frac{z}{z-1}$. It remains to be seen if these elements of Γ conjugate any $z \in \mathbb{C}$.

First, let $z \in \mathbb{C}$ such that $\frac{1}{z} = \bar{z}$. Thus, if $z = a + bi$,

$$\frac{1}{a+bi} = \left(\frac{1}{a+bi}\right)\left(\frac{a-bi}{a-bi}\right) = \frac{a-bi}{a^2+b^2}.$$

Thus, $\frac{1}{z} = \bar{z}$ if and only if $|z| = 1$, which is precisely C_1.

Now let $z \in \mathbb{C}$ such that $1 - z = \bar{z}$.

$$1 - (a + bi) = 1 - a - bi,$$

so $z = \bar{z}$ if and only if $a = 1 - a$, so we have that $a = 1/2$. Thus all such z have the form $1/2 + bi$, which is C_3.

Now let $z \in \mathbb{C}$ such that $\frac{z}{z-1} = \bar{z}$.

$$\frac{a+bi}{a+bi-1} = \frac{(a+bi)(a-1-bi)}{(a-1)^2+b^2} = \frac{a^2-a+b^2-bi}{(a-1)^2+b^2}.$$

For $\frac{z}{z-1} = \bar{z}$ to be true, it is necessary that $(a-1)^2 + b^2 = 1$, for otherwise $|Im(z)|$ would change, and the equality would be lost. Further calculation shows that $a = a^2 - a + b^2$ precisely when $(a-1)^2 + b^2 = 1$, which is C_2.

Now we know that if $f(z)$ is a degree 6 S_3-modular polynomial, one its roots, z_0, and its conjugate $\bar{z}_0$, must lie on $C_1, C_2, \text{or } C_3$. As the elements of Γ act with one orbit on the roots of $f(z)$ and permute $C_1, C_2, \text{and } C_3$, if one complex root is on one of the circles, all 6 roots must lie on the circles.

□

4. S_3 Modular Polynomials

We would like to be able to find all polynomials that are S_3-modular of weight 6. That is, we are looking for

$$P_k = \left\{ f(z) | deg(f(z)) \leq k, \text{and} (cz+d)^k f\left(\frac{az+b}{cz+d}\right) = f(z) \right\},$$

specifically when $deg(f(z)) = 6$. It is easier to first consider a closely related collection of functions.

Definition 6. *$F(x, y)$ is **homogeneous of degree** k if the sum of the powers of x and y is k. That is,*

$$F(x,y) = \sum_{i=0}^{k} a_i x^i y^{k-i}$$

[4].

We will first look at the homogeneous picture of the polynomials we are interested in. To do so we consider the following homogeneous analogues of the objects we are working with. Define $\widetilde{\Gamma}$ as

$$\widetilde{\Gamma} = \left\langle \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & -1 \\ 1 & -1 \end{bmatrix} \right\rangle.$$

$\widetilde{\Gamma}$ is a group with 12 elements, but as with $GL_2(\mathbb{Q})$, the elements are in positive and negative pairs. Thus $\widetilde{\Gamma}/\{\pm I\} \simeq S_3$.

$$\widetilde{P_k} = \left\{ F(x,y) | F \text{is homogeneous deg } k, F(ax+by, cx+dy) = F(x,y) \ \forall \ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \widetilde{\Gamma} \right\}.$$

Notice that if F is homogeneous of degree k it is possible to factor constants out of of the function:

$$F(\lambda x, \lambda y) = \sum_{i=0}^{k} a_i (\lambda x)^i (\lambda y)^{k-i} = \lambda^k \sum_{i=0}^{k} a_i x^i y^{k-i} = \lambda^k F(x,y).$$

Therefore, if $F(ax+by, cx+dy) = F(x,y)$, then

$$(cx+dy)^k F\left(\frac{ax+by}{cx+dy}, 1\right) = y^k F\left(\frac{x}{y}, 1\right).$$

Thus we have a relationship from $\widetilde{P_k} \to P_k$ defined by

$$F(x,y) \to F(z,1),$$

and from $P_k \to \widetilde{P_k}$ defined by

$$f(z) \to y^k f(\tfrac{x}{y}) = F(x,y).$$

Now we will look at two different polynomials from $\widetilde{P_3}$ and $\widetilde{P_2}$. Let $u = x^2 - xy + y^2$ and $v = xy(x-y)$. We will show that $u \in \widetilde{P_2}$ and $v \in \widetilde{P_3}$.

Proof. First we will explicitly state the elements of $\widetilde{\Gamma}$,

$$\widetilde{\Gamma} = \left\{ \pm \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \pm \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \pm \begin{bmatrix} -1 & 1 \\ -1 & 0 \end{bmatrix}, \pm \begin{bmatrix} -1 & 0 \\ -1 & 1 \end{bmatrix}, \pm \begin{bmatrix} 1 & -1 \\ 0 & -1 \end{bmatrix}, \pm \begin{bmatrix} 0 & -1 \\ 1 & -1 \end{bmatrix} \right\}.$$

Now consider $F(x,y) = u = x^2 - xy + y^2$. We will examine how the elements of $\widetilde{\Gamma}/\{\pm I\}$ effect u.

- $F(x,y) = x^2 - xy + y^2$
- $F(y,x) = y^2 - yx + x^2 = x^2 - xy + y^2$
- $F(x-y, -y) = x^2 - 2xy + y^2 + xy - y^2 + y^2 = x^2 - xy + y^2$
- $F(-x, y-x) = x^2 + xy - x^2 + y^2 - 2xy + x^2 = x^2 - xy + y^2$
- $F(-x+y, -x) = x^2 - 2xy + y^2 - x^2 + yx + x^2 = x^2 - xy + y^2$
- $F(-y, x-y) = y^2 + xy - y^2 + x^2 - 2xy + y^2 = x^2 - xy + y^2$.

Clearly the negatives of these elements of $\widetilde{\Gamma}$ act on u in the same manner. Since each element of $\widetilde{\Gamma}$ leaves u unaffected, $u \in \widetilde{P_k}$.

For v, we get the same results.

- $F(x,y) = xy(x-y)$
- $F(y,x) = yx(y-x) = xy(x-y)$
- $F(x-y, -y) = (y^2 - xy)(x - y + y) = xy(x-y)$
- $F(-x, y-x) = (x^2 - xy)(-x - y + x) = xy(x-y)$

- $F(-x+y,-x) = (x^2-xy)(-x+y+x) = xy(x-y)$
- $F(-y,x-y) = (y^2-xy)(-x) = xy(x-y)$.

The negatives of these elements of $\widetilde{\Gamma}$ act on v by changing the sign of the resulting function. The elements of $\widetilde{\Gamma}$ take v to v up to the sign, which is accounted for by the restriction by $\pm I$. ☐

Theorem 1. *$\widetilde{P_k}$ has a basis,*

$$\{u^i, v^j : 2i = 3j = k\}.$$

When $k=6$, this basis is of the form $\{u^3, v^2\}$.

The proof of this theorem is beyond the scope of this paper, but we examine a similar case in section 8.

Since there is a basis, all the polynomials in P_k in which $x=z$ and $y=1$ are of the form $f_a(z) = (z^2-z+1)^3 + a(z(z-1))^2$. We see now that the sign of v after being acted on by $\widetilde{\Gamma}$ is irrelevant since v is always raised to an even power.

Consider the family of polynomials

$$f_a(z) = (z^2-z+1)^3 + a(z^2-z)^2.$$

This family of polynomials fit the form for S_3-modular polynomials as generated above from the homogeneous case. We are specifically interested in f_a when its roots are complex.

Lemma 7. *$f_a(z)$ has complex roots if and only if $a > -27/4$.*

Proof. Let $u = z^2-z+1$. We can now write

$$f_a(z) = u^3 + a(u-1)^2.$$

We want to know for which values of a is $f_a(z)=0$ for some $z \in \mathbb{R}$. Observe that

$$f_a(z) = u^3 + a(u-1)^2 \Rightarrow -a = \frac{u^3}{(u-1)^2}.$$

It is clear that if the domain of u is $\mathbb{R}$, then a can take any value in $\mathbb{R}$. However, u is restricted. Simple calculations show that the minimum value of u is $3/4$, occurring at $z=1/2$, when $u-1=-1/4$. At this point, using the formula above, we see that

$$-a = \frac{(3/4)^3}{(-1/4)^2} = 27/4 \Rightarrow a = -27/4.$$

by merit of the fact that u^3 grows faster than $(u-1)^2$, $\frac{u^3}{(u-1)^2}$ attains its minimum on $[3/4,\infty)$ at $u=3/4$. Thus f_a has real roots for all $a \le -27/4$ and complex roots for all $a > -27/4$. ☐

A final point of interest is to see where these roots lie. Clearly as we vary a, the roots will migrate about the circles, and will do so continuously. If we start at $a=0$, there will be two triple roots at $1/2 \pm i\sqrt{3}/2$ where all three circles intersect. Notice that from each of the triple roots, there are six paths that the roots can follow along the circles. Now, if we let a move continuously in the negative direction, the three roots from $1/2+i\sqrt{3}/2$ will move directly towards the real axis on C_3, towards -1 on C_1 and towards 2 on C_2. Those from $1/2-i\sqrt{3}/2$ will be the conjugates of the first three roots. We can see this because when a reaches $-27/4$ there will be real roots, which obviously cannot happen if one of the roots moves along C_3 towards ∞. By taking the orbit of a point on the path towards ∞, we find the paths that our roots, which will become real, cannot take, so our roots must lie on the paths outside the orbit. If a were to vary in the positive direction, the roots will then move on the other three paths from each triple root, with one root moving towards ∞ from each triple root. This is justified by the same argument, with the realization that there will always be complex roots with a positive.

5. The Icosahedron

At this point we have generated a family of polynomials that are invariant with respect to a Möbius group that is isomorphic to S_3, one of the simplest nontrivial groups. It is our goal to use the techniques used in this smaller case to explore modular polynomials of a larger group.

The icosahedron is one of the five platonic solids. It is a regular polyhedron with equilateral triangles as faces, and five edges meeting at each vertex. It has 20 faces, 30 edges, and 12 vertices. For our purposes we are interested in the icosahedron that fits inside S^2, the 2-sphere, such that for each vertex v, $|v| = 1$. Just as S_3

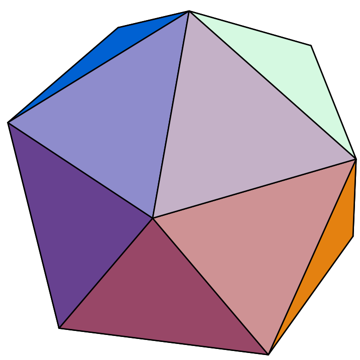

Figure 2. The Icosahedron.

is the symmetry group of the triangle, there is a group that corresponds to the symmetries of the icosahedron. To examine the symmetries of the icosahedron, we need only to look at its characteristic triangle. The characteristic triangle has three vertices, v_1, v_2, v_3 where v_1 corresponds to a vertex v, v_2 corresponds to the midpoint of an edge e, and v_3 corresponds to the centroid of a face f, and $v \in e \in f$. Any symmetry of the icosahedron must leave the relation between v_1, v_2 and v_3 unchanged. It is clear that any symmetry of the icosahedron will therefore be a rotation that sends the vertices to vertices, edges to edges, and faces to faces. Since the characteristic triangle must remain intact and not be reflected, all such symmetries will be rotations about vertices, the midpoints of edges or the centroids of faces. Clearly the rotation about a vertex will be by $k2\pi/5$, about the midpoint of an edge will be by $k\pi$, and about the centroid of a face by $k2\pi/3$, each with $k \in \mathbb{Z}$.

With this information we can generate the entire icosahedral group I. First there is the trivial symmetry of the identity. Then, with the axis of rotation passing through a pair of opposite vertices, there will be four nontrivial rotations by $k2\pi/5$, $k = 1, 2, 3, 4$. There are six distinct pairs of opposite vertices, so we have 24 more elements in I. For each of the fifteen distinct pairs of opposite edges there is one nontrivial rotation by π, adding 15 more elements to I. Similarly, the 10 distinct pairs of opposite centroids of faces have two nontrivial rotation by $k2\pi/3$, $k = 1, 2$, adding an additional 20 elements to I. This exhausts all the possible rotations about vertices, midpoints of edges and centroids of faces, so we can conclude that that $|I| = 60$.

It is of interest to us to describe the orbits of points on the icosahedron that arise from I.

Lemma 8. *Under the actions of I, the vertices of the icosahedron are a 12 element orbit, the midpoints of edges are a 30 element orbit, and the centroids of faces are a 20 element orbit. All other orbits have order 60.*

Proof. As the elements of I leave the characteristic triangle of the icosahedron invariant, the vertices are mapped to the vertices, midpoints to midpoints and centroids to centroids. Thus the vertices, midpoints and centroids from three distinct orbits, with 12, 30, and 20 elements respectively.

Now let x be a point on an edge of the icosahedron that is not a midpoint or endpoint. Then as each edge gets sent to every other edge under I, x is sent to 29 other points. However, as each edge has a corresponding rotation of π about its midpoint, each point on an edge that is not the midpoint gets mapped one other point on the edge. Thus x is mapped to the 29 other points as above, their 29 corresponding edge partners, and its own original edge partner. Thus the orbit of x has 60 elements.

Now let x be a point on a face of the icosahedron that is not the centroid, and not on an edge. As the elements of I map each face to all the others, there must be at least 19 more points in the orbit of x. As x is not the centroid of the face, the rotations about the centroid map x to two additional points on the face. Thus, there are three points on each face that x can be mapped to, placing 60 elements in the orbit of x.

□

For a given group G, an orbit is called **exceptional** if it has fewer than $|G|$ elements and **principal** if it has $|G|$ elements.

6. Generating the Icosahedral Group of Linear Fractional Transformations

We now know that we are looking for a Möbius group of order 60 that is isomorphic to I. This is a rather daunting task, but we can use the following construction taken from Toth [2]. First we must introduce some notation.

Notation 1. *For $\alpha \in S^2$ and $\theta \in \mathbb{R}$, let $R_{\theta,\alpha}$ denote the linear rotation with angle θ about the axis through $\pm\alpha$.*

Definition 7. *Let $\alpha \in S^2$. The* ***stereographic projection*** *$h(\alpha)$ is the point on the x, y plane that is collinear with α and $(0,0,1)$. Define $h((0,0,1)) = \infty$.*

Note that $h : S^2 \to \hat{\mathbb{C}}$, where $\hat{\mathbb{C}} = \mathbb{C} \cup \infty$ is the extended complex plane, is one to one and onto, so there will be a an inverse function $h^{-1} : \hat{\mathbb{C}} \to S^2$. We can explicitly define formulas for $h(\alpha)$ and $h^{-1}(z)$.

$$h(\alpha) = \frac{x_0 + ix_1}{1 - x_2}, \; \alpha = (x_0, x_1, x_2) \in S^2, \; \alpha \neq (0,0,1).$$

$$h^{-1}(z) = \left(\frac{2z}{|z|^2 + 1}, \frac{|z|^2 - 1}{|z|^2 + 1} \right) \in S^2, \; z \in \hat{\mathbb{C}}.$$

As defined, the formula for h^{-1} gives a complex number with norm less than or equal to 1, and the height off the complex plane, which is embedded in the x, y plane, where $h^{-1}(z)$ lives.

Note that we can stereographically project an image of the icosahedron onto $\hat{\mathbb{C}}$. First we project the edges of the icosahedron onto S_2, using the origin as the source of the projection. This results in a corresponding spherical tessellation, which we can then project onto $\hat{\mathbb{C}}$.

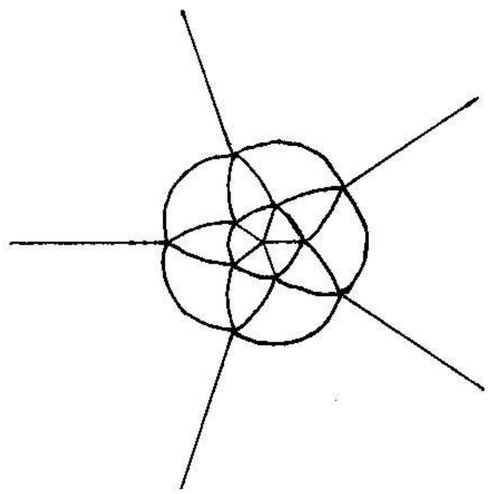

FIGURE 3. The stereographic projection of the icosahedron.

Now we can state an important theorem that will help construct the Möbius group for I.

Theorem 2. *Let R be the rotation $R_{\theta,\alpha}, \; \alpha \in S^2, \; \theta \in \mathbb{R}$, restricted to S^2. Then R conjugated with the stereographic projection h is the linear fractional transformation*

$$(h \circ R \circ h^{-1})(z) = \frac{xz - \bar{y}}{yz - \bar{x}}, z \in \hat{\mathbb{C}},$$

with

$$x = \cos\left(\frac{\theta}{2}\right) + i \sin\left(\frac{\theta}{2}\right) x_2,$$

and

$$y = \sin\left(\frac{\theta}{2}\right)(x_1 + ix_0).$$

The proof of the theorem is beyond the scope of this paper, but with the theorem we can easily generate the icosahedral group of linear fractional transformations.

The first step in the process is to find the location of the vertices of the icosahedron. We position the icosahedron such that one vertex lies at (0,0,1), and another at (0,0,-1). Thus, there is a pentagonal pyramid above the x, y plane with its apex at $(0, 0, 1)$ and its base parallel to the x, y plane, as well as a corresponding pentagonal pyramid inverted below the plane with $(0, 0, -1)$ as its apex. Since we are free to rotate the icosahedron about the z-axis without moving the vertices we have already fixed, we let the y-axis pass through the midpoint of an edge that connects a vertex from the upper pentagonal pyramid to one from the lower pentagonal pyramid.

We are already prepared to list some of the linear fractional transformations of I. Clearly there are the rotations about the z-axis, which are the rotations by $k2\pi/5$ for $k = 0, 1, 2, 3, 4$ about the origin. We will call these elements of I S^j, where $j = 0, 1, \ldots, 4$. Since rotation by θ about the origin is given by multiplying by an element $z \in \mathbb{C}$ such that $|z| = 1$ and $arg(z) = \theta$, we can explicitly state these rotations about the origin by

$$S^j : z \to \omega^j z,$$

where $j = 0, 1, \ldots, 4$, and ω is a fifth root of unity. A nth root of unity is an element $z \in \mathbb{C}$ such that $z^n = 1$. There are five fifth roots of unity, but we will choose $e^{i\frac{2\pi}{5}}$, since its powers generate all the fifth roots of unity.

Also, if we let $U = R_{\pi,(0,1,0)}$, the rotation about the midpoint of the edge which lies on the y-axis, and convert using the formulas above, we get $x = 0$ and $y = -1$, giving the linear fractional transformation

$$U : z \to -\frac{1}{z}.$$

Furthermore, we can compose U and S^j to generate 10 elements of I. Clearly we need to introduce more elements from I before we can generate the entire group through composition. To do so we will add two more axes passing though midpoints of edges. First add an axis that is perpendicular to the axis of U and which passes through the midpoint of an edge of the upper pentagonal pyramid. Call the rotation by π about this axis V. Notice that the axis of V is co-planar with the x-axis. Since U and V and perpendicular half turns, they commute, which the reader should verify. We will call their composition W. Since U and V commute we see that

$$W^2 = (UV)^2 = UVUV = UUVV = Id,$$

where Id is the identity. Thus W is another rotation by π about the midpoint of an edge, whose axis is perpendicular to those of U and V. Now we will generate the linear fractional transformations that correspond to U, V, and W.

Since we are no longer rotating about axes through obvious points, we must first list the vertices of the stereographic projection of the icosahedron. It is clear that two of the vertices, those at $(0, 0, 1)$ and $(0, 0, -1)$, correspond to ∞ and 0 respectively. It should also be apparent that the remaining ten vertices will fall on two circles centered at the origin, with the five upper pentagonal pyramid vertices falling on the larger circle and the five lower pentagonal pyramid vertices on the smaller circle. Furthermore, since these ten vertices are connected by equilateral triangles with edges that are parallel to the x, y plane, the projections of the upper vertices will fall between those of the lower vertices, that is, they will alternate. To find the norms of the projections, we must first examine the golden ratio.

Definition 8. *The ratio of the length of a segment connecting two non adjacent vertices of a regular pentagon and the length of an edge of the pentagon is defined to be ϕ. ϕ is commonly referred to as the **golden ratio**.*

Furthermore, a *golden rectangle* is a rectangle where the ratio of the sides is ϕ. Upon inspection, several golden rectangles can be found in the icosahedron. By connecting the endpoints of opposite edges in the logical way we get a golden rectangle, as the icosahedron edges are the sides of the bases of pentagonal pyramids. When two opposite edges are connected a diagonal is drawn across the base of a pentagonal pyramid, so the resulting rectangle must be golden.

By examining the golden rectangles in the icosahedron, we see that there are five with a corner at (0,0,1) and (0,0,-1). Picking any one of these we see that the other corners are conveniently on the bases of the upper and lower pentagonal pyramids. By looking directly at one of these rectangles we reduce the problem to two dimensions. Thus, by extending the edge connecting (0,0,1) and the vertex in the upper pyramid to the plane we find the magnitude of the projection of the vertex. Similarly, by seeing where the line connecting (0,0,1) and the lower vertex crosses the x, y plane we can find the magnitude of its projection. This calculation can be done by comparing similar triangles, and the magnitudes are ϕ for the upper vertices and $1/\phi$ for the lower vertices.

The five fifth roots of unity form a regular pentagon, giving the equality

$$\phi = \frac{|\omega - \omega^4|}{|\omega^2 - \omega^3|}.$$

Since both $\omega - \omega^4$ and $\omega^2 - \omega^3$ are purely imaginary we can simplify the equality to get

$$\phi = \frac{\omega - \omega^4}{\omega^2 - \omega^3}.$$

Noticing that $\omega^5 = 1$, we get $(\omega^2 - \omega^3)(\omega^2 - \omega^3) = -\omega^4 + \omega^6 = \omega - \omega^4$, and we now have the equality

$$\phi = -(\omega^2 + \omega^3).$$

Also, since $(\omega - \omega^4)(\omega + \omega^4) = \omega^2 - \omega^8 = \omega^2 - \omega^3$, we have

$$\frac{1}{\phi} = \omega + \omega^4.$$

Now we can express the stereographic projections of the vertices of the icosahedron purely in terms of ω. Since the upper and lower vertices alternate, we will start from $\omega + \omega^4$ and $\omega^2 + \omega^3$, generating the list

$$0,\ \infty,\ \omega^j(\omega + \omega^4),\ \omega^j(\omega + \omega^3), \quad j = 0, 1, 2, 3, 4.$$

Given this list of vertices, we can now generate the linear fractional transformations corresponding to U, V and W. As V is the rotation about the axis through the midpoint of an edge in the base of the upper pentagonal pyramid, and is coplanar with the x-axis, we can easily locate the edge midpoint in question. The midpoint must be in the outer ring of edges which corresponds to the base of the upper pyramid, and the x-axis must intersect that edge at its midpoint. This leaves only the the edge connecting $\omega^2(\omega^2+\omega^3)$ and $\omega^3(\omega^2+\omega^3)$. Thus to express V in $R_{\theta,\alpha}$ form, we have

$$\alpha' = \frac{1}{2}h^{-1}(\omega^2(\omega^2+\omega^3)) + \frac{1}{2}h^{-1}(\omega^3(\omega^2+\omega^3)).$$

Doing the computation, we see that

$$h^{-1}(\omega^2(\omega^2+\omega^3)) = \left(\frac{2(\omega^2(\omega^2+\omega^3))}{|(\omega^2(\omega^2+\omega^3))|^2+1}, \frac{|(\omega^2(\omega^2+\omega^3))|^2-1}{|(\omega^2(\omega^2+\omega^3))|^2+1}\right).$$

As $|(\omega^2(\omega^2+\omega^3))| = \phi$, we can simplify to get

$$h^{-1}(\omega^2(\omega^2+\omega^3)) = \left(\frac{2(\omega^2(\omega^2+\omega^3))}{\phi^2+1}, \frac{\phi^2-1}{\phi^2+1}\right).$$

Similarly we get

$$h^{-1}(\omega^3(\omega^2+\omega^3)) = \left(\frac{2(\omega^3(\omega^2+\omega^3))}{\phi^2+1}, \frac{\phi^2-1}{\phi^2+1}\right).$$

We already know that $\frac{1}{2}h^{-1}(\omega^2(\omega^2+\omega^3)) + \frac{1}{2}h^{-1}(\omega^3(\omega^2+\omega^3))$ must lie on the x-axis, so we know that when we express it in (x,y,z) form, the y value must be 0. Thus we have

$$\frac{1}{2}h^{-1}(\omega^2(\omega^2+\omega^3)) + \frac{1}{2}h^{-1}(\omega^3(\omega^2+\omega^3)) = \left(\frac{\omega^2(\omega^2+\omega^3)+\omega^3(\omega^2+\omega^3)}{\phi^2+1}, 0, \frac{\phi^2-1}{\phi^2+1}\right).$$

Now using the fact that the first coordinate is real we use the equality $\omega^2(\omega^2+\omega^3)+\omega^3(\omega^2+\omega^3) = (\omega^2+\omega^3)^2 = \phi^2$, and arrive at our final form

$$\alpha' = \frac{1}{2}h^{-1}(\omega^2(\omega^2+\omega^3)) + \frac{1}{2}h^{-1}(\omega^3(\omega^2+\omega^3)) = \left(\frac{\phi^2}{\phi^2+1}, 0, \frac{\phi^2-1}{\phi^2+1}\right).$$

However, we have not yet computed the α necessary to put the rotation into $R_{\theta,\alpha}$ form, since this α' does not live on S^2. By normalizing the vector we will find the necessary α which is on S^2 and shares the same axis through the origin. Straightforward calculations show that the normalized vector is

$$\alpha = \left(\frac{\phi}{\sqrt{\phi^2+1}}, 0, \frac{1}{\sqrt{\phi^2+1}}\right).$$

Now using α and $\theta = \pi$ we can compute the x and y of the linear fractional transformation, getting

$$x = \frac{i}{\sqrt{\phi^2+1}} \text{ and } y = \frac{i\phi}{\sqrt{\phi^2+1}}.$$

We can further simplify this x and y, using the fact that $\phi^2 = \phi+1$ to arrive at the equality

$$(-i(\omega-\omega^4))^2 = (\omega-\omega^4)^2 = \omega^2-2\omega^5+\omega^8 = \omega^2+\omega^3-2 = -\phi-2 = -(\phi+2) = -(\phi^2+1).$$

So by substituting $(-i(\omega-\omega^4))^2$ for $-(\phi^2+1)$ we have

$$x = \frac{1}{\omega-\omega^4} \text{ and } y = \frac{\omega^2+\omega^3}{\omega-\omega^4}.$$

It is worth noting that $\phi = \frac{1}{2}(1+\sqrt{5})$, which gives rise to another equality that we will use to simplify x and y one last time.

$$\phi + \frac{1}{\phi} = \frac{1+\sqrt{5}}{2} + \frac{2}{1+\sqrt{5}} = \frac{10+2\sqrt{5}}{2+2\sqrt{5}}.$$

Squaring gives

$$\left(\frac{5+\sqrt{5}}{1+\sqrt{5}}\right)^2 = \frac{30+10\sqrt{5}}{6+2\sqrt{5}} = 5,$$

so $\phi + 1/\phi = \sqrt{5}$. Applying this last equality, x and y simplify as follows.

$$x = -\frac{1}{\omega-\omega^4} = \left(\frac{1}{\omega-\omega^4}\right)\left(\frac{\omega^2-\omega^3}{\omega^2-\omega^3}\right) = -\frac{\omega^2-\omega^3}{\omega^2+\omega^3-(\omega+\omega^4)} = \frac{\omega^2-\omega^3}{\phi+1/\phi} = \frac{\omega^2-\omega^3}{\sqrt{5}},$$

and similarly

$$y = \frac{\omega-\omega^4}{\sqrt{5}}.$$

Now we are prepared to give the linear fractional transformation that corresponds to V as follows.

$$V : z \to \frac{(\omega^2-\omega^3)z+(\omega-\omega^4)}{(\omega-\omega^4)z-(\omega^2-\omega^3)}.$$

By composing V with U, we get

$$W : z \to \frac{-(\omega-\omega^4)z+(\omega^2-\omega^3)}{(\omega^2-\omega^3)z+(\omega-\omega^4)}.$$

Now we can finally state all the elements of the icosahedral group I by taking all the compositions available of U, V, W and the rotations about the z-axis.

$$I = \left\{\omega^j z, \frac{1}{\omega^j z}, \omega^j \frac{(\omega^2-\omega^3)\omega^l z+(\omega-\omega^4)}{(\omega-\omega^4)\omega^l z-(\omega^2-\omega^3)}, \omega^j \frac{-(\omega-\omega^4)\omega^l z+(\omega^2-\omega^3)}{(\omega^2-\omega^3)\omega^l z+(\omega-\omega^4)}\right\},$$

for $j, l = 0, 1, 2, 3, 4$. This is the complete icosahedral group.

7. Invariant Polynomials Under I

Now that we have the group I, we will find polynomials which are invariant under the actions of I. This construction is also taken from Toth [2]. There is an obvious choice for an invariant polynomial, since we know that the twelve vertices form an orbit under I, and we know the coordinates of their projections onto the plane. Thus, taking the homogeneous approach that we used in the S_3 case, we construct the following degree twelve homogeneous polynomial,

$$V(x,y) = xy\prod_{i=0}^{4}(x-\omega^i(\omega^2+\omega^3)y)\prod_{j=0}^{4}(x-\omega^j(\omega+\omega^4)y).$$

Since homogeneous coordinates will relate to affine coordinates by the relation $z = x/y$, the x and y at the beginning of the formula account for the vertices at 0 and ∞. The remaining terms in the product are the monomials for each of the ten remaining vertices.

As we saw earlier, terms involving ω tend to simplify nicely, and this is the case for V. Using various identities and the relations between the powers of ω and ϕ, V becomes

$$V(x,y) = xy(x^5-(\omega^2+\omega^3)^5y^5)(x^5-(\omega+\omega^4)^5y^5) = xy(x^5-\phi^5\omega^5)(x^5-\phi^{-5}\omega^5).$$

Finally, by applying the equality $\phi^5 - \phi^{-5} = 11$, V completely simplifies to

$$V(x,y) = xy(x^{10} + 11x^5y^5 - y^{10}).$$

We now possess a homogeneous polynomial that is invariant under I, since its roots are merely permuted by the elements of I. If a polynomial is invariant under a given Möbius group G, any G-invariant differential operator on that polynomial will result in another G-invariant polynomial. With this in mind we will use $Hess(V)(x,y)$ to generate another invariant polynomial, where the Hessian $Hess(\xi)$ of ξ, an invariant homogeneous polynomial for a Möbius group G, is defined as

$$Hess(\xi)(x,y) = \begin{bmatrix} \frac{\partial^2\xi}{\partial x^2} & \frac{\partial^2\xi}{\partial x\partial y} \\ \frac{\partial^2\xi}{\partial x\partial y} & \frac{\partial^2\xi}{\partial y^2} \end{bmatrix}.$$

$Hess(V)$ will be another G-invariant form which acts on the vector $\begin{bmatrix} x \\ y \end{bmatrix}$ by matrix multiplication. A straightforward calculation shows that

$$Hess(V)(x,y) = -121(x^{20} + y^{20}) + 27588(x^{15}y^5 - 494x^{10}y^{10} - x^5y^{15}).$$

Since multiplication by a scalar does not change the roots of an polynomial, we can normalize $Hess(V)$ to get

$$C(x,y) = -(x^{20} + y^{20}) + 228(x^{15}y^5 - 494x^{10}y^{10} - x^5y^{15}).$$

C is of degree 20, so it has 20 roots which are permuted by the elements of I. Thus the roots of C must be the projections of the centroids of the faces of the icosahedron, as they are the only orbit of order 20.

Another G-invariant differential operator is the Jacobian $Jac(\xi_0, \xi_1)$, where ξ_0 and ξ_1 are invariant homogeneous polynomials G, defined as

$$Jac(\xi_0, \xi_1) = \begin{bmatrix} \frac{\partial\xi_0}{\partial x} & \frac{\partial\xi_0}{\partial y} \\ \frac{\partial\xi_1}{\partial x} & \frac{\partial\xi_1}{\partial y} \end{bmatrix}.$$

Taking $\xi_0 = V$ and $\xi_1 = C$ we obtain

$$Jac(V,C) = 20(x^{30} + y^{30}) + 10440(x^{25}y^5 - x^5y^{25}) - 200100(x^{20}y^{10}x^{10}y^{20}),$$

and by normalizing,

$$M(x,y) = (x^{30} + y^{30}) + 522(x^{25}y^5 - x^5y^{25}) - 10005(x^{20}y^{10}x^{10}y^{20}).$$

As M has degree 30, its roots must be the projections of the midpoints of the edges of the icosahedron, as there are no other orbits of degree 30.

These three invariant polynomials correspond to the three exceptional orbits of I, and all other I-invariant polynomials can be expressed as polynomials of V, C and M. Computation also shows that V, C and M are linearly dependent. This follows from the following equality.

$$1728V^5 - M^2 - C^3 = 0.$$

8. Icosahedral Polynomials of Weight 60

Definition 9. *We will call any polynomial that is invariant under the actions of I an **icosahedral polynomial**.*

We know that any polynomial which is invariant under the actions of I can be written as a polynomial of V, C and M. This is justified by the fact that V, C and M represent the only non-principal orbits on the icosahedron. Suppose there is another invariant form A. The roots of A may contain an exceptional orbit, in which case A is divisible by V, C or M. Otherwise, the roots of A are a principal orbit, and we can generate a complex ratio μ such that $C^3 + \mu V^5$ will vanish at some point in that orbit. (Notice that we do not necessarily need to relate V and C. Any combination of the three will work when raised to the necessary powers that result in an order 60 polynomial.) If this polynomial vanishes at one point on the orbit, that orbit must be the roots of the polynomial. Thus we see that any invariant form can be expressed by an algebraic combination of V, C and M.

We are particularly interested in polynomials of degree 60, since such polynomials would have 60 roots which would correspond nicely to an orbit on the icosahedron. The three obvious examples V^5, C^3 and M^2 are all examples of invariant polynomials on I, but they are rather mundane as they have repeated roots at the vertices, centroids of the faces and midpoints of the edges respectively. Clearly a polynomial with 60 distinct roots is much more interesting.

Just as the S_3 case has a basis $\{u^i, v^j : 2i = 3j = k\}$, the invariant polynomials for I will also have a basis in some combination of V, C and M. We would naively like to think that it is $\{V^i, C^j, M^k : 12i = 20j = 30k = n\}$ for a constant n, but since V, C and M are algebraically dependent, this cannot be a basis. Using the substitution $M^2 = 1728V^5 - C^3$, the basis becomes

$$\{V^i, C^j : 12i = 20j = n\}.$$

Using the basis above, and realizing that we are interested in the polynomials of V and C up to scalar multiplication, we can derive the general form for a degree 60 invariant homogeneous polynomial with respect to I:

$$F(x, y) = (C(x, y))^3 + a(V(x, y))^5,$$

for a constant a. We can now change the homogeneous polynomials to standard polynomials through the mapping $F(x, y) \to F(z, 1)$ so that

$$V(x, y) \to F_v(z) = z(z^{10} + 11z^5 - 1),$$

and

$$C(x, y) \to F_c(z) = -(z^{20} + 1) + 228(z^{15} - 494z^{10} - z^5).$$

Now it is clear that the constant a must go before the V term since it is only of degree 11 in affine coordinates, whereas C remains of degree 20. We are then assured that any polynomial we generate will have degree 60. Now we can explicitly state the general form of all degree 60 icosahedral polynomials in monic form:

$$((z^{20} + 1) - 228(z^{15} - 494z^{10} - z^5))^3 + a(z(z^{10} + 11z^5 - 1))^5.$$

As with the S_3 case, we are interested in where the roots of these polynomials lie. We took an algebraic approach before to find the points that are conjugated by the elements of S_3, but now it will be easier to take a geometric approach.

Using the projection of the icosahedron from above, see that complex conjugation is equivalent to reflecting about a face of the icosahedron. Thus, the roots of icosahedral polynomials will be points that, when reflected about a face, remain in their I orbit. Clearly this is true for any vertex, midpoint of an edge, centroid of a face or point on an edge. However, there are many more points on faces that will qualify. The points that lie on the axes of mirror symmetry are precisely those points that will not leave their orbits when reflected about a face. Thus, on each face there will be three lines connecting vertices and the midpoints of the opposite edges that give the desired points.

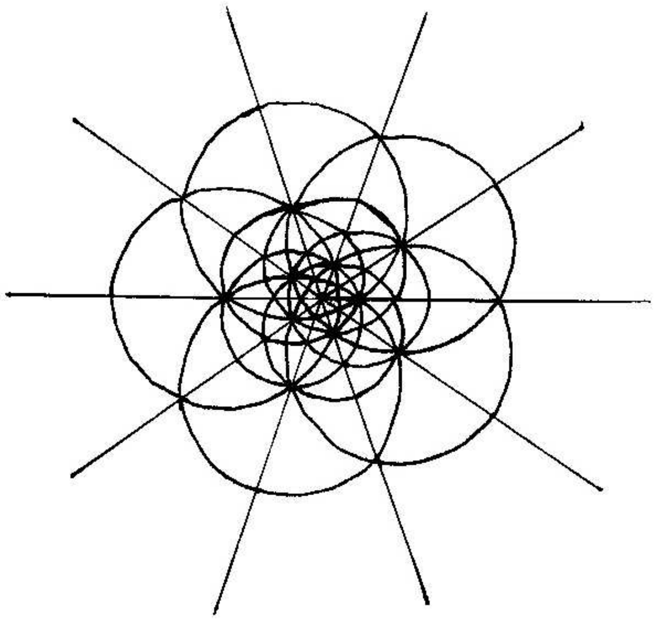

FIGURE 4. The roots of the degree 60 icosahedral polynomials.

Now we have a beautiful analogue to the S_3 case. Remember that there are three trivial icosahedral polynomials of degree 60: one with double roots at the midpoints of edges, one with triple roots at the centroids of faces, and one with quintuple roots at the vertices. Just as when we varied a in the S_3 case the roots moved continuously along the circles, they will do so in the icosahedral case. Suppose we start with the triple roots at the centroids. This happens for $a = 0$. As we vary a, three roots emerge from the triple roots and either move towards the three vertices of the face or the midpoints of the three edges of the face. Since we get the double roots at the midpoints when $a = -1728$, we see that as a moves in the negative direction from 0, the roots migrate towards the midpoints. As a passes -1728 in in the negative direction, two roots meet at each midpoint, and then proceed along the edges in opposite directions towards the adjacent vertices. Notice that if a is positive the roots also move towards the vertices from the centroids. This is expected since the vertices include the point at ∞, which can be approached from from both the positive and negative directions.

Now we make the final observation that the projection of the roots of the icosahedral polynomials is the same as the projection of the one of the Catalan solids, the disdyakis triacontahedron, which is alternately known as the hexaxis icosahedron [3]. The disdyakis triacontahedron is composed of one triangle and its mirror image, has 120 faces, 180 edges, 12 vertices of degree 10, 20 vertices of degree 6 and 30 vertices of degree 4. This is clearly equivalent to our icosahedron with the subdivided faces. As the disdyakis triacontahedron has the same symmetry group as the icosahedron, we can conclude that the orbits of the roots of the icosahedral polynomials correspond to the orbits of the edge points of the disdyakis triacontahedron.

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