Stability of Splay States in Coupled Oscillator Networks

Author: Amy Lynn Nesky

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Amy Nesky

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1 Abstract

There are countless occurrences of oscillating systems in nature. Climate cycles and planetary orbits are a few that humans experience daily. Man has also incorporated, to his benefit, oscillation into his craft; the grandfather clock, for example, can keep track of time with astounding accuracy using the period of a long pendulum.

Such systems can range in complexity in a number of ways. The governing equation for a given oscillator could be as simple as a sine curve, or its motion could appear so erratic that oscillatory motion is undetectable to viewers. The number of oscillators in a system can also vary, and oscillators can be coupled; that is, oscillators can be affected by the motion of neighboring oscillators. It is this last case we wish to study.

An example is two pendulums linked by a spring (*Figure 1*):

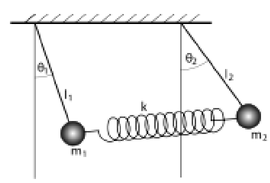

Figure 1: Coupled pendulums [2].

We will briefly look at the case of finitely many oscillators and then move to analyzing a model consisting of infinitely many identical oscillators. Synchrony is the simplest collective behavior. We will study a more complicated pattern called *splay states* in which oscillators are equally staggered in phase, i.e. phase locked such that the system will return to this pattern if it is disturbed by an arbitrarily small amount. Mathematically, this requires us to find attracting fixed points in the system. We will approximate the local behavior of our model by linearizing the system near its fixed points. We will then apply our findings to a few specific cases of such models including: uniform density, linear distribution, α-function pulses, and integrate-and-fire.

2 Introduction

Perhaps at first, the idea of many oscillators entering a synchronous state sounds far fetched, but it is no more far fetched than planetary motion. In fact, there are numerous natural occurrences of such systems. One breed of fireflies native to parts of Southeast Asia provides concrete evidence to support the possibility of such synchronous states arising. Fireflies light up for an assortment of reasons: as a warning sign to predators, to distinguish between species of fireflies, to distinguish between gender and as a mating call [1]. These Southeast Asian Fireflies gather by the thousands in mangrove trees at dusk. At the beginning of the evening they flash independently of one another, but after a short time, and without maestro, the fireflies begin to flash in unison. Some theorize that this is a mating call meant to be visible to ladies through thick forestation, but regardless of the motivation, the sight is spectacular. "Imagine a tree thirty-five to forty feet high..., apparently with a firefly on every leaf and all the fireflies flashing in perfect unison at the rate of about three times in two seconds, the tree being in complete darkness between flasshes .... Imagine a tenth of a mile of river front with an unbroken line of trees with fireflies on every leaf flashing in synchronism, the insects on the trees at the ends of the line acting in perfect unison with those between," writes H.M. Smith [4].

This example is not unique. Other biological examples include the networks of neurons in circadian pacemaker and in the hippocampus, crickets that chirp in unison, populations of women whose menstral cycles become synchronized, the pacemaker cells of the heart and the cells of the pancreas that secrete insulin [3]. Mutual synchrony is less implausible than perhaps initially thought. Since the grandfather clock is such a hit, it seems a worthy cause to examine the underlying mathematics that govern such mutual synchrony and potentially harness a whole new technological tool.

Analyzing only a few oscillators is difficult; analyzing thousands, or in our case infinitely many, proves extremely challenging. One of the first to tackle this problem was Arthur T. Winfree, who discovered that mutual synchronization is a cooperative event [6]. Renato E. Mirollo and Steven H. Strogatz picked up where Winfree left off and studied Peskin's model of the cardiac pacemaker, that is a model in which a population consists of idential integrate-and-fire oscillators with pulsatile coupling [3]. They discovered that for almost all initial conditions in this model, a synchronous state emerges.

Before Mirollo and Strogatz, most of the work done on mutual synchronization assumed that coupling between oscillators was smooth; that is, that the oscillators could always *see* each other so to speak. One can compare smooth coupling to runners running around a race track - the runners can see each others' positions in their respective cycles at all times. An integrate-and-fire oscillator is one in which energy is gathered and emitted in a pulse. One could liken this model to a race in which the runners are blindfolded as they make their way around the track such that their only interaction with one another takes place when a runner crosses the lap line, at which point a whistle is blown to inform the other runners that one runner has completed a lap. In this model, when one oscillator fires, or emits a pulse, the other oscillators in the population either jump ahead in their cycle by a fixed amount (i.e. gain a fixed amount of energy) or reach their own firing threshold, whichever would occur sooner (i.e. requires less energy). The firefly example is most accurately represented by this model, because the only communication between the insects occurs when one firefly emits a flash. The human pacemaker is also best understood with Peskin's model since the cells that make up the pacemaker only interact when a cell discharges a pulse.

In this paper we study a general framework for an identical oscillator network, linearize this system near its fixed points and apply our results to a this special case of infinitely many integrate-and-fire oscillators with pulsatile coupling as well as the case of a uniform density population, $\Phi(x) = x$.

3 Finite Identical Oscillator Network

In finite identical oscillator networks, each oscillator evolves independently with respect to this coupling, while the global current or impulse if driven collectively by the whole network. These equations are derived from Josephson Junctions. The following two equations describe a continuous coupling:

$$\dot{x}_j = f(x_j) + KM \qquad \text{and} \qquad A\ddot{M} + B\dot{M} + CM = \frac{1}{n}\sum_{j=1}^{n} g(x_j).$$

The following variation describes pulsitile coupling:

$$A\ddot{M} + B\dot{M} + CM = \frac{1}{n}\sum_{j=1}^{n} \delta(t - t_j), \qquad \text{where } A, B, C > 0, \text{ and } t_j, j \in [1, n] \text{ are previous firing times.}$$

At $t = t_j$, we add to the existing $M(t)$ the function $m(t)$ which satisfies:

$$m(t_j) = 0$$
$$m'(t_j) = \frac{1}{An}.$$

$m(t)$ represents the energy distribution to all of the n oscillators in the when one oscillator fires; i.e., when one oscillator releases a pulse all of the oscillators in the network are stimulated in this fashion. We know,

$$M(t_j - \epsilon) = M(t_j + \epsilon)$$
$$M'(t_j + \epsilon) = \frac{a}{An} + M'(t_j - \epsilon).$$

Recall, the δ "function" is defined by:

$$\delta(x) = 0 \qquad \forall x \neq 0$$
$$\int_{-\infty}^{\infty} \delta(x)\ dx = \int_{-\epsilon}^{\epsilon} \delta(x)\ dx = 1.$$

Thus, we have,

$$\int_{t_j-\epsilon}^{t_j+\epsilon} \delta(t - t_j)\ dt = 1.$$

And,

$$\int_{t_j-\epsilon}^{t_j+\epsilon} A\ddot{M} + B\dot{M} + CM\ dt = A[\dot{M}(t_j+\epsilon) - \dot{M}(t_j-\epsilon)] + B[M(t_j+\epsilon) - M(t_j-\epsilon)] \qquad \because M(t_j-\epsilon) = M(t_j+\epsilon).$$

4 Continuum Limit of Identical Oscillators

Now we let $n \to \infty$, and examine a "continuum limit," identical oscillator network. $\rho(x)$ will be the density function of the oscillators such that $\rho(\mathrm{x}) \geq 0$ on $[0, 1]$. The equations governing the evolution of $\rho(x)$, and M are as follows:

$$\dot{x} = f(x) + KM$$
$$A\ddot{M} + B\dot{M} + CM = (f(0) + KM)\rho(0), \qquad \text{where } (f(0) + KM)\rho(0) \text{ is the flux across the threshold.}$$

We need the boundary condition:

$$(f(1) + KM)\rho(1) = (f(0) + KM)\rho(0).$$

Thus,

$$\nu(x) = (f(x) + KM)\rho(x) \text{ is continuous on } S^1.$$

The evolution of desities in this model is governed by the Fokker-Plank equation, which states

$$\dot{\rho}(x) + [(f(x) + KM)\rho(x)]' = 0.$$

In stationary states (densities) $(\rho(x), M)$ of the infinite case, $\dot{\rho}(x) = 0$. We make the change of variables $s = KM$ and we consider s as parameterizing the splay states. Then by the Fokker-Plank equation, we have,

$$(f(x) + s)\rho(x) = c \qquad \text{for some contant } c$$
$$\implies \rho(x) = \frac{c}{f(x) + s}$$
$$\implies c = \left(\int_0^1 \frac{1}{f(x) + s}\ dx \right)^{-1}.$$

We know $\dot{x} = f(x) + s$ has period:

$$T = \int_0^1 \frac{1}{f(x) + s}\ dx.$$

Thus $c = \frac{1}{T}$, and we can say

$$\begin{aligned} M &= C^{-1}(f(0)+s)\rho(0) = C^{-1}(f(0)+s)\frac{1}{T}\frac{1}{f(0)+s} \qquad \because \ddot{M} = \dot{M} = 0 \text{ in a stationary state} \\ &= (CT)^{-1} \\ \implies K &= CTs. \end{aligned} \tag{1}$$

Now we wish to linearize the system near stationary densities to determine whether they are stable. We will do this by perturbing the system slightly; i.e., let

$$\begin{aligned} \tilde{\rho}(x) &= \rho(x) + \epsilon\eta(x) \\ \tilde{M}(x) &= M + \epsilon N \end{aligned}$$

where η, N depend on t, and $\rho(x)$ and M are both fixed in time (because we are considering stationary densities). Then the evolution of η and N is given by the equations

$$[\rho(x)+\epsilon\eta(x)]^{\cdot} + [(f(x)+K(M+\epsilon N))(\rho(x)+\epsilon\eta(x))]' = 0 \tag{2}$$

$$A(M+\epsilon N)^{\cdot\cdot} + B(M+\epsilon N)^{\cdot} + C(M+\epsilon N) = [f(0)+K(M+\epsilon N)]\cdot[\rho(0)+\epsilon\eta(0)]. \tag{3}$$

Equation (2) becomes

$$\epsilon\dot{\eta}(x) + (f(x)+K(M+\epsilon N))(\rho'(x)+\epsilon\eta'(x)) + (f'(x)+K(M'+\epsilon N'))(\rho(x)+\epsilon\eta(x)) = 0$$

since we know $\rho(x)$ is fixed in time. Gathering the ϵ terms and ignoring the ϵ^2 terms since these are insignificant we find

$$\epsilon\dot{\eta}(x) + [(f(x)+s)\rho(x)]' + \epsilon[(f(x)+s)\eta(x)]' + \epsilon K\rho'(x)N = 0.$$

But $[(f(x)+KM)\rho(x)]' = 0$ as shown above. So the evolution of η is given by

$$\dot{\eta}(x) + [(f(x)+s)\eta(x)]' + K\rho'(x)N = 0.$$

Equation (3) becomes

$$\begin{aligned} & A(M+\epsilon N)^{\cdot\cdot} + B(M+\epsilon N)^{\cdot} + C(M+\epsilon N) = [f(0)+K(M+\epsilon N)]\cdot[\rho(0)+\epsilon\eta(0)] \\ \implies & (A\ddot{M}+B\dot{M}+CM) + \epsilon[A\ddot{N}+B\dot{N}+CN] = (f(0)+s)\rho(0) + \epsilon[(f(0)+s)\eta(0)+K\rho(0)N] \\ & \implies A\ddot{N}+B\dot{N}+CN = (f(0)+s)\eta(0)+K\rho(0)N \\ & \qquad \because A\ddot{M}+B\dot{M}+CM = (f(0)+s)\rho(0). \end{aligned}$$

Thus, the equations of the evolution of $\eta(x)$ and N are

$$\begin{aligned} \dot{\eta}(x) &= -[(f(x)+s)\eta(x)]' - K\rho'(x)N \\ A\ddot{N} + B\dot{N} + CN &= (f(0)+s)\eta(0) + K\rho(0)N. \end{aligned}$$

The next step in linearizing the system near stationary densities is to replace $\eta(x)$ by $e^{\lambda t}\eta(x)$ and N by $e^{\lambda t}N$, such that $e^{\lambda t}$ absorbs all time dependence in $\eta(x)$ and N. Then we have

$$[e^{\lambda t}\eta(x)]^{\cdot} = -[(f(x)+s)e^{\lambda t}\eta(x)]' - K\rho'(x)e^{\lambda t}N \tag{4}$$

$$A(e^{\lambda t}N)^{\cdot\cdot} + B(e^{\lambda t}N)^{\cdot} + C(e^{\lambda t}N) = (f(0)+s)e^{\lambda t}\eta(0) + K\rho(0)e^{\lambda t}N. \tag{5}$$

Looking at equations (4), we find

$$\begin{aligned} [e^{\lambda t}\eta(x)]^{\cdot} &= -[(f(x)+s)e^{\lambda t}\eta(x)]' - K\rho'(x)e^{\lambda t}N \\ \implies \lambda e^{\lambda t}\eta(x) &= -e^{\lambda t}[(f(x)+s)\eta(x) + K\rho(x)N]' \\ \implies \lambda\eta(x) &= -[(f(x)+s)\eta(x)]' - K\rho'(x)N, \end{aligned}$$

and at equation (5) we see

$$\begin{aligned} A(e^{\lambda t}N)^{\cdot\cdot} + B(e^{\lambda t}N)^{\cdot} + C(e^{\lambda t}N) &= (f(0)+s)e^{\lambda t}\eta(0) + K\rho(0)e^{\lambda t}N \\ \implies e^{\lambda t}(A\lambda^2 + B\lambda + C)N &= e^{\lambda t}[(f(0)+s)\eta(0) + K\rho(0)N] \\ \implies (A\lambda^2 + B\lambda + C)N &= (f(0)+s)\eta(0) + K\rho(0)N. \end{aligned}$$

Together this gives

$$\lambda\eta(x) = -[(f(x)+s)\eta(x)]' - K\rho'(x)N \tag{6}$$

$$q(\lambda)N = (f(0)+s)\eta(0) + K\rho(0)N, \tag{7}$$

where $q(\lambda) = A\lambda^2 + B\lambda + C$. Since this system is linear in η and N, we can assume that $N = 0$ or 1. We now define $\nu(x) := \big(f(x)+s\big)\eta(x)$, which means $\eta(x) = T\rho(x)\nu(x)$. Suppose for a moment that $N = 0$, and consider equation (6), then we have

$$\nu'(x) + \lambda T\rho(x)\nu(x) = 0 \text{ and } \nu(x)|_{x=0} = 0 \implies \nu(x) = 0 \ \ \forall x.$$

Since $N = 0$ has only the trivial solution, we can assume without loss of generality that $N = 1$:

$$\begin{aligned} \lambda\eta(x) &= -[(f(x)+s)\eta(x)]' - K\rho'(x) \\ q(\lambda) &= (f(0)+s)\eta(0) + K\rho(0). \end{aligned}$$

We can rewrite these equations (6) and (7) as

$$\begin{aligned} \nu'(x) + [\lambda T \rho(x)]\nu(x) &= -K\rho'(x) \\ q(\lambda) &= \nu(0) + K\rho(0) \end{aligned}$$

where $\nu(x) = (f(x)+s)\eta(x)$ as before. We can simplify this system further by defining $\mu(x) := \nu(x)+K\rho(x)$; then equations (6) and (7) become

$$\begin{aligned} \mu'(x) + [\lambda T \rho(x)]\mu(x) &= \lambda T K \rho^2(x) \\ q(\lambda) &= \mu(0). \end{aligned} \tag{8}$$

Let us consider for a moment the second change of variables. We multiply equation (8) by an appropriate integrating factor such that

$$I(x)\mu'(x) + I(x)[\lambda T \rho(x)]\mu(x) = I(x)\lambda T K \rho^2(x)$$

where,

$$I'(x) = I(x)[\lambda T \rho(x)] \implies I(x) = e^{\lambda T \int_0^x \rho(t)\ dt} = e^{\lambda T \Phi(x)} \qquad \text{and } \Phi(x) = \int_0^x \rho(t)\ dt.$$

Observe that $\Phi(x)$ is the cumulative distribution function for the density $\rho(x)$, so we have $\Phi(0) = 0$ and $\Phi(1) = 1$. Then we have

$$\begin{aligned} &[I(x)\mu(x)]' = I(x)\lambda T K \rho^2(x) \\ \implies &I(x)\mu(x) = c + \lambda T K \int_0^x I(t)\rho^2(t)\ dt \qquad \text{for some constant } c \\ \implies &\mu(x) = I(x)^{-1}\left[c + \lambda T K \int_0^x I(t)\rho^2(t)\ dt\right]. \end{aligned}$$

Observe that $\mu(0) = c$, so the equation $q(\lambda) = \mu(0)$ implies $q(\lambda) = c$. We now consider the boundary condition obtained from $q(\lambda) = \mu(0)$, that is

$$\mu(x)\Big|_{x=0}^{x=1} = 0.$$

This implies

$$\begin{aligned} I(1)^{-1}\left[c + \lambda T K \int_0^1 I(t)\rho^2(t)\ dt\right] - cI(0)^{-1} = 0 \\ \implies \lambda T K \int_0^1 I(t)\rho^2(t)\ dt = c(e^{\lambda T} - 1). \end{aligned}$$

Thus, we find the characteristic equation in λ:

$$
\begin{aligned}
q(\lambda) &= \mu(0) = c \\
\implies q(\lambda)\frac{e^{\lambda T}-1}{\lambda T} &= K\int_0^1 I(t)\rho^2(t)\ dt. \qquad (9)
\end{aligned}
$$

Notice here that both sides of the above equation depend on λ since $I(x) = e^{\lambda T\Phi(x)}$.

5 Case *f(x)=a* For Constant *a*

Now we have the ability to look at particular cases. The case of a uniform density of oscillators, for example, has results that are easily obtained. In the case, $\dot{x} = f(x) = a$ for some constant a, $\Phi(x) = x$, and we have $I(t) = e^{\lambda T\Phi(x)} = e^{\lambda Tx}$, $\rho^2(t) = 1$, and

$$T = \int_0^1 \frac{dx}{a+s} = \frac{1}{a+s}.$$

Then

$$I(x) = e^{\lambda T\Phi(x)} = e^{\lambda Tx} = \exp\left(\frac{\lambda x}{a+s}\right),$$

and equation (9) gives the characteristic equation:

$$
\begin{aligned}
q(\lambda)\frac{e^{\lambda T}-1}{\lambda T} &= K\int_0^1 e^{\lambda Tt}\ dt \\
\implies (q(\lambda)-K)(e^{\lambda T}-1) &= 0 \qquad\qquad \text{for } \lambda \neq 0.
\end{aligned}
$$

This equation in λ has zeros at $A\lambda^2 + B\lambda + C = K$ and at $\lambda T = 2\pi i n$ for $n \neq 0$. Since $A, B, C > 0$, we find that both roots of $q(\lambda) - K = 0$ have negative real part. Let λ_a and λ_b be the roots of $0 = A\lambda^2 + B\lambda + C - K$, then we know,

$$\lambda_a + \lambda_b = \frac{-B}{A} < 0 \qquad \text{and} \qquad \lambda_a\lambda_b = \frac{C-K}{A}$$

We recall

$$
\begin{aligned}
K &= CTs \\
\implies \frac{K}{C} &= Ts = \int_0^1 \frac{s}{f(x)+s}dx < 1 \\
\implies C - K &> 0 \qquad (10) \\
\implies \lambda_a\lambda_b &> 0
\end{aligned}
$$

If $\lambda_a, \lambda_b \in \mathbb{R}$ then

$$
\begin{aligned}
\lambda_a\lambda_b > 0 &\implies \lambda_a > 0 \text{ and } \lambda_b > 0\text{, or } \lambda_a < 0 \text{ and } \lambda_b < 0 \\
\lambda_a + \lambda_b < 0 &\implies \lambda_a < 0 \text{ and } \lambda_b < 0
\end{aligned}
$$

if $\lambda_a, \lambda_b \in \mathbb{C}$ then,

$$\lambda_{a,b} = \frac{-B}{2A} \pm \frac{\sqrt{B^2 - 4A(C-K)}}{2A} \text{ have real part } \frac{-B}{2A} < 0$$

Let λ_n be the roots of $\lambda T = 2\pi i n$ such that $\lambda_n T = 2\pi i n$ $\forall n \in \mathbb{Z}$. Then we have $\lambda_n = 2\pi i(a+s)n$. *Figure 2* depicts the roots of $(q(\lambda) - K)(e^{\lambda T} - 1) = 0$ in the complex plane.

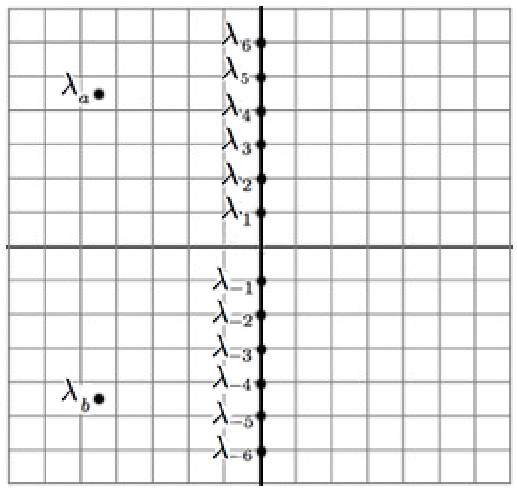

Figure 2: Roots of system governed by $f(x) = a$.

6 Case *f(x)=a-bx* For Constant *a* and *b*

One case of special interest to us is a network governed by

$$\dot{x} = f(x) = a - bx, \qquad \text{for some contants } a, b.$$

which has general solution $x(t) = -ke^{-t} + a/b$ for some constant k. Solving for T we find

$$T = \int_0^1 \frac{dx}{f(x)+s} = \int_0^1 \frac{dx}{a - bx + s} = \frac{\log(a+s) - \log(a+s-b)}{b}.$$

Continuing on we find

$$\begin{aligned}\Phi(x) &= \frac{1}{T}\int_0^x \frac{1}{f(t)+s}\,dt \\ &= \frac{b}{\log(a+s) - \log(a+s-b)}\int_0^x \frac{1}{a - bt + s}\,dt \\ &= \frac{\log(a+s) - \log(a+s-bx)}{\log(a+s) - \log(a+s-b)}.\end{aligned}$$

We note that $\Phi(0) = 0$, and $\Phi(1) = 1$, as expected. Hence

$$I(x) = e^{\lambda T \Phi(x)} = \exp\left[\frac{\lambda}{b}\log\left(\frac{a+s}{a+s-bx}\right)\right] = \left(\frac{a+s}{a+s-bx}\right)^{\lambda/b}, \tag{11}$$

and the characteristic equation is

$$\begin{aligned} q(\lambda)\frac{e^{\lambda T}-1}{\lambda T} &= K\int_0^1 \left(\frac{a+s}{a+s-bt}\right)^{\lambda/b}\left(\frac{1}{T^2}\right)\left(\frac{1}{a+s-bt}\right)^2 dt \\ &= \frac{K}{T^2}(a+s)^{\lambda/b}\int_0^1 (a+s-bt)^{-(\lambda/b+2)}\,dt, \end{aligned}$$

by equation (9). We can simplify this to

$$\begin{aligned} q(\lambda)\frac{e^{\lambda T}-1}{\lambda} &= Cs(a+s)^{\lambda/b}\int_0^1 (a+s-bt)^{-(\lambda/b+2)}\,dt \qquad \because K = CTs \text{ by eq. (1)} \\ &= Cs(a+s)^{\lambda/b}\left[\frac{(a+s-b)^{-(\lambda/b+1)}-(a+s)^{-(\lambda/b+1)}}{\lambda+b}\right] \\ &= \frac{Cs}{\lambda+b}\left[\frac{1}{a+s-b}\cdot\left(\frac{a+s}{a+s-b}\right)^{\lambda/b} - \frac{1}{a+s}\right] \\ \implies q(\lambda)e^{\lambda T} - q(\lambda) &= \frac{Cs\lambda}{\lambda+b}\left[\frac{1}{a+s-b}\cdot e^{\lambda T} - \frac{1}{a+s}\right] \qquad \because e^{\lambda T} = \left(\frac{a+s}{a+s-b}\right)^{\lambda/b} \text{ by eq. (11)} \\ \implies q(\lambda) - \frac{Cs\lambda}{\lambda+b}\cdot\frac{1}{a+s} &= e^{\lambda T}\left[q(\lambda) - \frac{Cs\lambda}{\lambda+b}\cdot\frac{1}{a+s-b}\right]. \end{aligned}$$

Solving for $e^{\lambda T}$ gives

$$\begin{aligned} e^{\lambda T} &= \frac{q(\lambda) - \frac{Cs\lambda}{(\lambda+b)(a+s)}}{q(\lambda) - \frac{Cs\lambda}{(\lambda+b)(a+s-b)}} \\ &= 1 + \frac{b}{(\lambda+b)(a+s)(a+s-b)}\cdot\frac{Cs\lambda(\lambda+b)(a+s-b)}{q(\lambda)(\lambda+b)(a+s-b)-Cs\lambda} \qquad \text{for } q(\lambda) - \frac{Cs\lambda}{(\lambda+b)(a+s-b)} \neq 0 \\ &= 1 + \frac{b}{(a+s)}\cdot\frac{Cs\lambda}{q(\lambda)(\lambda+b)(a+s-b)-Cs\lambda}. \end{aligned} \tag{12}$$

7 Relationship between *f(x)=a* and *f(x)=a-bx* For Constants *a,b*

We will now to find a perturbation relationship between the eigenvalues of the system governed by $f(x) = a$ and those of the system governed by $f(x) = a - bx$. To do this we consider T and λ_n to be functions of b. If $f(x) = a - bx$ governs the oscillators, we saw that $T(b) = \frac{1}{b}\log\left(\frac{a+s}{a+s-b}\right)$. We note that

$$\lim_{b\to 0} T(b) = \lim_{b\to 0}\frac{\log(a+s)-\log(a+s-b)}{b} = \lim_{b\to 0}\frac{\frac{1}{a+s-b}}{1} = \frac{1}{a+s} = T(0) \qquad \text{(by L'Hôpital's Rule)},$$

which matches the case of $f(x) = a$, as we expected. Similarly we note,

$$e^{\lambda T}\Big|_{b=0} = 1 + \frac{b}{(a+s)} \cdot \frac{Cs\lambda}{q(\lambda)(\lambda+b)(a+s-b) - Cs\lambda}\Bigg|_{b=0} = 1,$$

which yeilds the familiar equation $e^{\lambda T(0)} - 1 = 0$ as found the case of $f(x) = a$, and

$$q(\lambda) - \frac{Cs\lambda}{(\lambda+b)(a+s-b)}\Bigg|_{b=0} = q(\lambda) - CT(0)s = q(\lambda) - K.$$

Recall that the roots $\lambda_n(0) = (2\pi in)/T(0) = 2\pi in(a+s)$ are the eigenvalues when the system is governed by $f(x) = a$. We will describe the eigenvalues when the system is governed by $f(x) = a - bx$ as perturbations of the eigenvalues of the system governed by $f(x) = a$. Let $\lambda_n(b)$ be an eigenvalue of the system $f(x) = a - bx$. Using the Taylor series expansion we write

$$\lambda_n(b) = \lambda_n(0) + \delta_n b + (\text{higher order terms in } b) \qquad \text{for } n \in \mathbb{N}.$$

When b is sufficiently small, the first order estimate will be

$$\lambda_n(b) \approx \frac{2\pi in}{T(0)} + \delta_n b.$$

In other words, we hope to find a δ_n such that $\lambda_n(b)$ satisfies equation (12), which describes the system for $f(x) = a - bx$. We will now approximate the left and right hand sides of equation (12) to first order using $\lambda = \lambda_n(b)$ for small b. We first look at the left hand side. Using the Maclaurin series of $log(1+x)$, we find

$$\begin{aligned}
T(b) = \frac{1}{b}\log\left(1 + \frac{b}{a+s-b}\right) &\approx \frac{1}{b}\left[\frac{b}{a+s-b} - \frac{1}{2}\left(\frac{b}{a+s-b}\right)^2\right] \\
&= \frac{1}{a+s-b} - \frac{b}{2(a+s-b)^2} = \frac{2(a+s)-3b}{2(a+s-b)^2} \\
\implies T'(b) &\approx \frac{-6(a+s-b)^2 + 4(2(a+s)-3b)(a+s-b)}{4(a+s-b)^4} \\
\implies T'(0) &= \frac{1}{2(a+s)^2}.
\end{aligned}$$

So the Maclaurin series of T is

$$T(b) \approx T(0) + bT'(0) \approx \frac{1}{a+s} + \frac{b}{2(a+s)^2} = \frac{2(a+s)+b}{2(a+s)^2}.$$

Then

$$
\begin{aligned}
\lambda_n(b)\cdot T(b) &\approx 2\pi i n + \left[T(0)\delta_n + \frac{2\pi i n}{T(0)}T'(0)\right]b \\
\implies e^{\lambda_n(b)\cdot T(b)} &\approx \exp\left(T(0)\delta_n b + \frac{2\pi i n}{T(0)}T'(0)b\right) \\
&\approx 1 + T(0)\delta_n b + \frac{2\pi i n}{T(0)}T'(0)b \\
&\approx 1 + \frac{\delta_n b}{a+s} + 2\pi i n(a+s)\frac{1}{2(a+s)^2}b \\
&\approx 1 + \frac{(\delta_n + \pi i n)b}{a+s}.
\end{aligned}
$$

The right hand side of equation (12) to first order in b is

$$
\begin{aligned}
1 + \frac{b}{(a+s)}\cdot\frac{Cs\lambda_n(0)}{q(\lambda_n(0))(\lambda_n(0)+b)(a+s-b) - Cs\lambda_n(0)} &= 1 + CsT(0)\cdot\frac{b\lambda_n(0)}{q(\lambda_n(0))(\lambda_n(0)+b)(a+s-b) - Cs\lambda_n(0)} \\
&\approx 1 + K\cdot\frac{b\lambda_n(0)}{\frac{q(\lambda_n(0))(\lambda_n(0))}{T(0)} - Cs\lambda_n(0)} \qquad \text{by eq. (1) and } \because b \approx 0 \\
&= 1 + K\cdot\frac{bT(0)}{q(\lambda_n(0)) - CsT(0)} \\
&= 1 + bT(0)\cdot\frac{K}{q(\lambda_n(0)) - K}.
\end{aligned}
$$

Thus

$$
\begin{aligned}
\frac{(\delta_n + \pi i n)b}{a+s} &= bT(0)\cdot\frac{K}{q(\lambda_n(0)) - K} \\
\implies \delta_n &= \frac{K}{A(\lambda_n(0))^2 + B(\lambda_n(0)) + C - K} - \pi i n \\
&= \frac{K}{[-A(2\pi n(a+s))^2 + C - K] + i[B2\pi n(a+s)]} - \pi i n \\
&= \frac{K\big[[-A(2\pi n(a+s))^2 + C - K] - i[B2\pi n(a+s)]\big]}{[A(2\pi n(a+s))^2 + C - K]^2 + [B2\pi n(a+s)]^2} - \pi i n.
\end{aligned}
$$

Thus, for small b, the eigenvalues of the system governed by $f(x) = a - bx$ can be thought of as perturbations of the eigenvalues of the system governed by $f(x) = a$ by the amount δ_n above. To determine the stability of $\lambda_n(b)$ for small b, we consider the sign of $Re(\delta_n)$, since $\lambda_n(0)$ has neutral stability as show in *Figure 2* . We have

$$
Re(\delta_n) = \frac{K[-A(2\pi n(a+s))^2 + C - K]}{[A(2\pi n(a+s))^2 + C - K]^2 + [B2\pi n(a+s)]^2}.
$$

Note that equation (10) says that $C - K > 0$. It is sufficient to consider δ_1 since $Re(\delta_1) < 0$ implies that $Re(\delta_n) < 0$ for all $n \in \mathbb{N}$, and since δ_n and $\delta_{-n}$are conjugate pairs for all $n \in \mathbb{N}$. We see

$$Re(\delta_1) = \frac{K[-A(2\pi(a+s))^2 + C - K]}{[A(2\pi(a+s))^2 + C - K]^2 + [B2\pi(a+s)]^2}.$$

We recall,

$$\begin{aligned} K &= \frac{Cs}{a+s} \qquad \text{by eq. (1)} \\ &= C\left(1 - \frac{a}{a+s}\right) \\ \implies 1 - \frac{K}{C} &= \frac{a}{a+s} \\ \implies \frac{Ca}{C-K} &= a+s. \end{aligned}$$

Thus, $Re(\delta_1) < 0$ if and only if

$$\begin{aligned} 0 &> K[-A(2\pi(a+s))^2 + C - K] \\ \iff K - C &> -A(2\pi\frac{Ca}{C-K})^2 \\ \iff (C-K)^3 &< 4\pi^2 a^2 A C^2. \end{aligned}$$

We know $0 < K < C$, so we get stability as $K \to C^-$. In the limit as $K \to 0^+$, we would need

$$\begin{aligned} C^3 &< 4\pi^2 a^2 A C^2 \\ \iff C &< 4\pi^2 a^2 A \end{aligned}$$

in order to have stability. If this holds, we get stability for all $K > 0$. If $C > 4\pi^2 a^2 A$, then there exists a critical $0 < K_c < C$ satisfying $(C - K_c)^3 = 4\pi^2 a^2 A C^2$ such that for $K < K_c$ the splay is unstable and for $K > K_c$ the splay is stable.

8 Special Case: α-Function Pulses

The condition $C < 4\pi^2 a^2 A$ has a simple physics interpretation in the special case where pulses are α-functions, i.e. where the system is governed by:

$$\ddot{M} + 2\alpha\dot{M} + \alpha^2 M = 0.$$

This will be our previous case with $C/A = \alpha^2$ and $B/A = 2\alpha$. The characteristic equation of this differential equation is $(\lambda+\alpha)^2 = 0$; i.e., $\lambda = -\alpha$ is a double root. A special solution is of the form $M(t) = te^{-\alpha t}$. Thus, $\dot{M}(t) = (1 - \alpha t)e^{-\alpha t}$. Here $\tau = 1/\alpha$ is a time constant that measures the width of the pulse.

The critical case then becomes

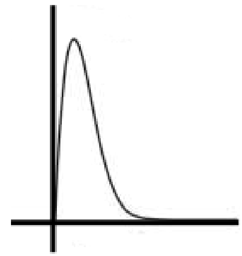

Figure 3: Pulse function.

$$\alpha^2 < 4\pi^2 a^2$$
$$\implies \tau > \frac{1}{2\pi a}.$$

We note that $1/a$ is aprroximately the period of the unforced oscillator in the system. Thus we get stability for all $K > 0$ when

$$\alpha < 2\pi a.$$

9 Special Case: Integrate-and-Fire

Another case of special interest to us is a network governed by

$$\dot{x} = f(x) = a - x \qquad \text{for some contant } a,$$

which has general solution $x(t) = -ke^{-t} + a$ for some constant k. For simplicity, we will take firing threshold to be $x(t) = 1$ such that a given oscillator will pulse and reset when $x(t) = 1$ (see *Figure 4*). This case however is simply the previous case, in which $f(x) = a - bx$, with $b = 1$.

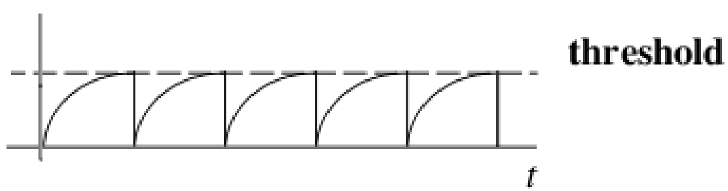

Figure 4: Integrate-and-fire.

10 Conclusion

We have discovered that the stability of these splays is a subtle thing that depends on the shape of the pulse relative to the period of the oscillators. We did a first order calculation, so our conclusions are valid for b sufficiently small, which we never made precise. The next step in this investigation would be to try to find out what happens to the eigenvalues $\lambda_n(b)$ for the case governed by $f(x) = a - bx$ for larger values of b and what happens to the eigenvalues λ_a and λ_b for the case governed by $f(x) = a$, as depicted in *Figure 2*, when the system is perturbed by various values of b.

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