Brownian Motion: A Study of Its Theory and Applications

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A Study of Its Theory and Applications

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I

An Introduction to Brownian Motion

The theory of Brownian motion is an integral part of statistics and probability, and it also has some of the most diverse applications found in any topic in mathematics. With extensions into fields as vast and different as economics, biology, and management science, Brownian motion has become one of the most studied mathematical phenomena of the late twentieth and early twenty-first centuries. Today, Brownian motion is mostly understood as a type of mathematical process called a stochastic process. The word "stochastic" actually stems from the Greek verb *στοχάζεςθαι* (or *στοχάζομαι*), which translates as "I guess" or "I aim," implying that stochastic processes tend to produce uncertain results, and Brownian motion is no exception to this, though with the right models, probabilities can be assigned to certain outcomes and we can begin to understand these complicated processes (Taylor 2).

Mathematically, Brownian motion can be thought of as a continuous time process in which over every infinitely small time interval Δt, the entity under consideration moves one "step" in a certain direction. This interpretation suggests that Brownian motion can be viewed as a "random walk" process, and this is one key aspect of Brownian motion that will be discussed. In reality, however, there is a lot more to Brownian motion than just its randomness, because as the word "walk" implies, this process must lead to some end, and this will be discussed as well (Ross 452). One of the most interesting aspects of Brownian motion, along with stochastic processes in general, is that from the randomness that characterizes the process, we can make fairly accurate predictions about how the process will behave later. This theme is essential to probability and statistics, so it will be revisited frequently, garnering the attention that it requires.

Pollen Contributes to Something Other Than Allergies

The physical manifestation of Brownian motion was most famously observed by, not surprisingly, a Scottish botanist named Robert Brown in 1827. His interpretation of this process was based on the movement of small pollen particles suspended in a drop of water. In his experiments, the pollen particles appeared to move in a completely random fashion, stumping Brown and his colleagues. Upon further investigation, Brown, among others, verified that this phenomenon was not unique to pollen particles, but rather was exhibited by many different types of microscopic particles suspended in a fluid. During this time it was also discovered that this process depended on several variables, including the fineness of the particles, whether or not heat was introduced into the system, and the viscosity of the fluid medium (Taylor 473).

Interestingly, Brown did not actually comprehend what he was witnessing immediately upon observation. Initially, he thought that he had made a landmark discovery of the essence of a life form's ability to live and grow, but once it became known that pollen was not the only substance that exhibited this behavior, he quickly abandoned this claim (Lemons 17). Brown was not the first person to observe the random motion of microscopic particles, but he was the first to carefully lay out his observations and begin to question why this process took place (even though some of his theories were off base). Because of this, it is fair to say that Robert Brown provided the motivation for the development of the kinetic theory of heat and helped set the table for one of

Einstein's landmark 1905 papers, "On the Motion of Small Particles at Rest Required by the Molecular-Kinetic Theory of Heat," in which he goes beyond Brown's general observations of Brownian motion and establishes the physical theory driving this natural phenomenon (Stachel 75).

A Genius Fails to Disappoint

Brown's initial observations of Brownian motion went almost eighty years without any significant mathematical or even physical explanation. Enter Albert Einstein. In 1905 alone, Einstein published more significant scientific papers (five) than many scientists put forth over the course of their entire lives (Stachel 4). His foray into the molecular-kinetic theory of heat is sometimes overshadowed by his other more famous discoveries, such as the Theory of Relativity and the Photoelectric Effect, but it seems that Brownian motion, with its plentiful applications, can be considered one of Einstein's most productive discoveries (Satchel 73).

Revisiting the puzzle of the randomly moving pollen grains, Einstein claimed that the motion of these microscopic particles stemmed from the constant forces exerted on the particles from the surrounding fluid, thanks to individual fluid molecules bumping into the particles, thus sending them in motion. Since the pollen grains were completely surrounded by these fluid molecules, they experience these forces in every conceivable direction, which explains why the particles do not move in a set pattern or direction. This explanation may seem mundane by today's standards, considering that much of modern science is intently focused on happenings on microscopic scales, but in 1905, this finding was certainly not trivial. At this time, knowledge of even the most basic elementary particles was in its infancy, so when Einstein took something as well-established as Brownian motion and explained it soundly using these new ideas, it certainly drew worldwide attention from the physics community (Taylor 474).

As it turns out, though, as brilliant as his take on the molecular-kinetic theory of heat was, Einstein did not fully grasp the magnitude of his findings until after he published his paper. Einstein later admitted that he wrote this work, “without knowing that observations concerning Brownian motion were already long familiar” (Stachel 78), which is remarkable, considering that this topic had been somewhat unresolved for many years. It then makes sense as to why the title of the paper does not contain the words “Brownian motion” and why in the actual paper he only mentions Brownian motion as being identical to the motion he is describing. It is incredible that Einstein was somehow in the dark with regards to the issue of Brownian motion, but it is even more incredible that despite this lack of information he was still able to produce a work that greatly helped develop a problem he was relatively unfamiliar with (Stachel 78).

The actual mathematical discussion of Einstein’s paper, however, is dated. Einstein’s mathematics cater more to the physical phenomenon of Brownian motion, whereas the actual development of Brownian motion as a stochastic process did not surface until 1923, when Norbert Weiner, a mathematician at MIT at the time (Jerison 432), established the modern mathematical framework of what is known today as the Brownian motion random process. This is why Brownian motion is sometimes referred to as the Wiener process or even the Wiener-Einstein process. In fact, it would make more sense if this process were known by one of these two names, considering Einstein and Weiner contributed much more to this theory than Brown ever did. This is especially true because the study of Brownian motion today mostly involves the stochastic process pioneered by Weiner, rather than the physical process studied by Brown (Taylor 474).

The Math Behind the Madness

Unfortunately, understanding the historical context surrounding the Brownian motion stochastic process does not help one understand the complicated mathematics that it entails. In order to do this, it is necessary to start with the basic ideas of stochastic processes, of which Brownian motion is one type. A stochastic process is most simply understood as a set of random variables, and is often denoted as $X(t)$, where t belongs to T, the index set of the process. For the purposes of Brownian motion, t usually denotes time taken from an index set containing some continuous time interval. The random variable $X(t)$ is known as the state of the stochastic process at time t, and this is the main component of the process that is of interest. There are many different types of stochastic processes, of which Brownian motion is only one. Brownian motion, however, can be grouped with a much larger classification of stochastic processes, and by understanding these processes it is possible to begin to understand the math behind Brownian motion (Ross 73-74).

These stochastic processes are called Markov chains, and they are absolutely essential in understanding Brownian motion, among other physical processes. Markov chains are stochastic processes that deal with the probability of the process changing from one specific state to another. Imagine a stochastic process that ranges over a finite or countable infinite number of outcomes. Usually, this range of outcomes is represented by the non-negative integers (i.e. 0, 1, 2, …). These values are known as the state of the process at a time n, which is represented by $X_n = i$, where i is the state. Whenever the process is in a state i, there is a definite probability P_{ij} that it will next be in state j. In terms of a conditional probability, we have

$$P_{ij} = P\{X_{n+1} = j \mid X_n = i, X_{n-1} = i_{n-1}, \ldots, X_1 = i_1, X_0 = i_0\}$$

What this equation says is that the next state of the Markov chain depends only on the present state of the chain, and not on any of the past states. Being able to ignore past states is helpful when trying to figure out what happens next in the process. Kolmogorov's probability theory also tells us that for a Markov chain making a transition from a state i into a state j (Larsen 37),

$$P_{ij} \geq 0, \quad i, j \geq 0, \sum_{\text{all } j} P_{ij} = 1, i = 0, 1, 2, \ldots$$

Now that the basic properties of Markov chains have been presented, it makes sense to reinforce this knowledge with an example (Ross 137).

The Math of Sin

One of the most common applications of Markov chains is the Gambler's Ruin problem. The problem itself is described by a Markov chain where X_n is the gambler's "bankroll" at a time n and N is the gambler's target winnings. So, this means that $P_{00} = P_{NN} = 1$, because if the gambler has nothing to wager, he is forever stuck at zero and has no chance of accumulating any more money. A similar reasoning applies for P_{NN}, because if the gambler sits at N, his desired total, he technically will not wager anymore, so he will stay at N with probability 1 (until he decides to play again; he is a gambler after all). These recurrent states are trivial, however, when looking at the entire problem in perspective. What really interests us are the transient states of the problem, where we try to figure out the probability of whether the gambler will achieve his goal or go broke.

Say that for this particular problem, at each point during the game the gambler has probability p of winning one unit and probability $q = 1 - p$ of losing one unit. The ultimate outcome, then, obviously depends directly on p. But is this all his success depends on? The answer is no, because how much is in his bankroll also determines his fate in this game of chance. Let P_i, $i = 0, 1, \ldots, N$, be the gambler's probability of attaining his goal N starting with i units. Assuming first that N is infinite, it can be shown that

$$P_i = 1 - (q/p)^i, \quad p > 1/2$$

$$P_i = 0, \qquad p \leq 1/2$$

This means that if the gambler were to play this game forever without stopping, unless his probability of winning on a given trial is greater than half, the gambler will go broke eventually with probability 1. What about for a finite N? Then the equation becomes modified a little bit (Ross 164).

$$P_i = [1 - (q/p)^i]/[1 - (q/p)^N], \quad p \neq 1/2$$

$$P_i = i/N, \qquad p = 1/2$$

So, for most versions of this game (where the probability would most likely be stacked in favor of the house), the first equation would apply. For a version of the game with equal probability of winning and losing (like flipping coins), though, the second equation would apply.

In order to better grasp the chances of winning a game like this, it makes sense to plug some numbers into these equations. Say you are playing horseshoes with a friend and you decide to wager on each toss you make. You decide that whoever lands closer to the stake on each throw wins one dollar from the other person. This continues until one person has won all of the money. To start, you only have 10 dollars in your wallet and your friend has 25 dollars. You are slightly better at horseshoes, however, so on any one toss you have a probability of .65 of landing closer to the stake. What is the probability that you eventually end up with all of his money?

We can use the equation $P_i = [1 - (q/p)^i]/[1 - (q/p)^N]$ since p is not equal to 1/2 and N is finite. Plugging in $p = .65$, $q = 1 - p = .35$, $i = 10$ and $N = 35$ (total money in play), we see that $P = .998$, so it is almost a certainty that you will clean your friend out after a certain number of tosses. Say you only started with one dollar, however. Would your superior skill be enough to overcome this deficit? Using $i = 1$, we see that $P = .462$, a significantly lower probability, but not as low as one may expect. Even starting on the brink of bankruptcy, you still have almost a 50/50 chance of winning all the money. When the probability is in your favor, like many other gambling games, the results can be very lucrative. Unfortunately, they call it the "Gambler's Ruin" problem for a reason, so it is assumed that more often than not the probability is not in your favor, implying that you lose more than you win (Ross 162-165).

Continuing on to Continuous Time

Talking about discrete-time cases is helpful and informative, but most relevant stochastic processes (i.e. ones that have applications to the real world) are classified by continuous-time states. Continuous-time Markov chains are very similar to Markov chains as described above in that they possess the essential property that the future behavior of the process depends only on the present state and not on any previous state.

This property is very helpful when attempting to work with continuous-time Markov chains, because these processes are even more complicated than the discrete-time version. Consider a continuous-time process $X(t)$, which is defined for all $t \geq 0$, that ranges over the non-negative integers. We can call this process $X(t)$ a continuous-time Markov chain if for all values $s, t \geq 0$ and integers $i, j, x(u) \geq 0, 0 \leq u \leq s$

$$P\{ X(t + s) = j \mid X(s) = i, X(u) = x(u), 0 \leq u \leq s \} = P\{ X(t + s) = j \mid X(s) = i \}$$

What this equation states is that regardless of a state $x(u)$ that the process was in at an earlier time u, the state of the process at a future time $t + s$ only depends on the current state i, found at time s (Ross 249).

An additional property of continuous-time Markov chains that is common in real-world systems is the idea of having stationary probabilities. This means that the transition of a Markov chain to its next state is independent of the current time s, which implies that no matter how long a process remains in a certain state, the process never is forced to make a transition based on the time spent in only one state. For example, imagine that a continuous-time Markov chain enters a state at a time t, and stays there for m minutes. Then the probability that the process remains in the same state for the next n minutes depends only on n, not on t or m. In mathematical notation, with the random variable X_i representing the time spent in the same state I, we have

$$P\{ X_i > m + n \mid X_i > m \} = P\{ X_i > n \}$$

for all $m, n \geq 0$. This implies that X_i is memoryless and has an exponential distribution, a distribution usually used to predict "waiting time" (Ross 203).

Some Final Thoughts on Markov Chains

The necessary mathematical prerequisites of Brownian motion have been laid out, but before going on some other ideas need to be established. The Gambler's Ruin problem, as described above, is an example of a random walk, which is a special type of Markov chain. A random walk models the probability of a process taking one "step" either to the right or left at a certain point in time. Using notation, it is easy to see that

$$P_{i,\, i+1} = p = 1 - P_{i,\, i-1}$$

where $i = 0, \pm 1, \pm 2, \ldots$and p is the probability that the process moves one step to the right from any state i. This implies that the probability of the process moving one step to the left from any i is $1 - p$ (Ross 137). The only difference between the Gambler's Ruin and a general random walk is that there are no such things as absorbent states in the general case (i.e. a bankroll of zero). It will be shown that Brownian motion is mostly concerned with processes that resemble the Gambler's Ruin problem, because in the real world there is little practicality in dealing with infinite states. One example of this idea is stock prices in that they cannot possess a negative value, so obviously zero is an absorbent state for a model predicting the future prices of a stock. This specific application will be dealt with later, because naturally before applications of Brownian motion can be discussed it is necessary to talk about Brownian motion itself.

II

Brownian Motion Defined and Explored

This Is It

Now that Markov chains and random walks have been discussed, it is possible to finally define what is known as the Brownian motion stochastic process. Imagine a Markov chain where $P_{i, i+1} = P_{i, i-1} = 1/2$, for all i belonging to the set of integers. This Markov chain is indeed a random walk, but because it has an equal chance of going either in one direction or the other, it is called a symmetric random walk. In addition, this symmetric random walk is defined by infinitesimal time increments (Δt) in which the steps become infinitesimal in measure (Δx). This random walk is a little different in that it no longer deals with steps of one unit, but rather steps of Δx units. This is an important property because it allows the Brownian motion process to be treated as a continuous function of t in which the change in x is without discontinuities. This function of t will simply be denoted as $X(t)$, where

$$X(t) = \Delta x(X_1 + X_2 + \dots + X_{[t/\Delta t]})$$

The random variables X_i correspond to the direction of the step at that point of the process. If the ith iteration of the process moves to the right, then $X_i = +1$, and if it moves to the left, $X_i = -1$. Summing the values of each X_i of the process up until t and multiplying by Δx gives the net displacement of the process at time t. Something that is not obvious from just looking at the equation is how many steps of the process there are from the beginning of the process to time t (Ross 452). It is easy to see that there are $[t/\Delta t]$ random variables, but what does this number mean? The brackets refer to a special function called the greatest integer function. This function produces the largest integer that is less than or equal to the number in question, here the ratio $t/\Delta t$ (Weisstein). This number $t/\Delta t$ takes the time t and divides it into increments of length Δt, so hence the total number of random variables in the expression for $X(t)$ is this ratio.

Since the probability of each X_i moving in one direction is equal to the probability of it moving in the other direction, it is easy to see that the expectance of X_i, $E(X_i)$, is equal to zero. It follows that

$$Var(X_i) = E(X_i^2) - E(X_i)^2 = 1 - 0 = 1$$

since $E(X_i^2)$ is equal to one because X_i^2 is equal to one for all i, since $X_i = \pm 1$. Extending this idea to the Brownian motion process $X(t)$ and using the summing properties of expectation and variance, we see that $E[X(t)] = 0$ and $Var[X(t)]$ equals the sum of the variances of the $[t/\Delta t]$ X_i's multiplied by the square of Δx, treated as a constant in this situation, or $Var[X(t)] = (\Delta x)^2[t/\Delta t]$.

What is really of interest when looking at this process is how it behaves when its increments occur at instantaneous times, so now it is necessary to consider the above expressions as Δt approaches zero. The expression for the variance needs to be handled carefully, however, because if we just let Δt go to zero, then the variance becomes infinite, and this should not be. To counter this, we let Δx approach zero as well, but in a way that connects it with Δt. By considering $\Delta x = c\sqrt{\Delta t}$, where c is a constant, we see that the Δt terms cancel each other out of the expression, which leaves the limits of the expectation and variance as Δt approaches zero as

$$E[X(t)] = 0 \text{ and } Var[X(t)] = c^2 t$$

These expressions are very manageable, and they help define Brownian motion (Ross 452-453). The existence of the constant c suggests that the Brownian motion process is actually a family of processes connected to a specific parameter c, so to simplify things we will mostly consider the case where $c = 1$, which is known as the Standard Brownian Motion process (Ross 454).

Answer Me These Questions Three

Now that the basic principles of Brownian motion have been defined, it is logical to move on and explore the various properties of this process. There are three major properties that are attributed to Brownian motion, and they essential in defining how Brownian motion behaves. First, and most obviously, a standard Brownian motion process $X(t)$ is distributed normally with mean equal to zero and variance equal to t. This fact is apparent since $X(t)$ is the sum of many independent random variables, so the Central Limit Theorem tells us that the addition of these random variables results in a normal distribution.

Secondly, since the steps of the process take place over independent intervals, it is true that $X(t)$ has what are known as independent increments. This means that the intervals between the random variables are independent random variables as well. So, for all times t_i of $X(t)$,

$$X(t_i) - X(t_{i-1}),\ X(t_{i+1}) - X(t_i)$$

are independent. The independent increments assumption allows us to sum the variances, as in the previous section.

Finally, the third major property of Brownian motion is that the motion of the process depends only on the length of the interval, and not on the time t that the interval begins at. Due to this property it is said that Brownian motion also has stationary increments, and more precisely this means that the distribution of $X(t+s) - X(t)$ depends only on s, and not what happened up to and including the starting point t. This arises from the stationary probabilities assumption of Markov chains, and it makes sense in the context of random walks because a random walk would not be random at all if it had any sort of deterministic quality to it (Ross 453).

On a deeper mathematical level, the stationary and independent increments of Brownian motion tie directly into the fact that the variance of the process is a linear function of t (namely t itself, in many cases). Consider the increment $X(s + t) - X(0)$, where $t < s$. By the independent increments assumption, this is equal to $[X(t) - X(0)] + [X(s + t) - X(t)]$, and by the stationary increments assumption, we know that the latter half of the expression depends only on the duration of the interval, s. Taking the variances of both sides, this leaves us with the expression

$$Var[X(s + t)] = Var[X(s)] + Var[X(t)]$$

and the only way an equation like this could be solved is if the variances were functions of time, otherwise the variances would not sum in this way (Taylor 478).

To summarize, a stochastic process $X(t)$ is a Brownian motion process if it satisfies these three conditions:

a. $X(t)$ follows a normal distribution with $\mu = 0$ and $\sigma^2 = t$

b. $X(t)$ has independent increments

c. $X(t)$ has stationary increments

Another possible property of Brownian motion is that it starts at the origin (i.e. $X(0) = 0$), but this is not a necessary condition for the functionality of Brownian motion. Using this fact often makes calculations easier, so for this reason this property will be understood as holding for all Brownian motion processes, even though this might not always be the case. For example, $X(0)$ could equal some other arbitrary value, say m, and the only thing that would change is the mean, which would just shift by m (Ross 478).

It's a Hit

An aspect of Brownian motion that is of much interest is the expected time for a specific process to reach a predetermined value. Such information could be very useful for those who use Brownian motion to model real world situations. Let T_a be the point in time at which the process first reaches a, which is some pre-selected value that can either be greater than or less than zero. First, let $a > 0$. The expression of interest to compute is $P\{T_a \leq t\}$, which gives the probability that T_a, known as the hitting time, is less than some known time t. In order to get this expression, we need to consider $P\{X(t) \geq a\}$ while taking into account its dependence on whether or not $T_a \leq t$. This makes sense, because in order for $T_a \leq t$, $X(t)$ must be greater than a at the same time t. So, using conditional probability formulas, we can expand $P\{X(t) \geq a\}$as

$$P\{X(t) \geq a \mid T_a \leq t\}P\{T_a \leq t\} + P\{X(t) \geq a \mid T_a > t\}P\{T_a > t\}$$

The first part of this expression covers the case when $T_a \leq t$ and the second part conditions on if $T_a > t$. This half is obviously equal to zero, however, because if $T_a > t$, then the process has not yet reached a and $P\{X(t) \geq a\}$must be zero. So, we only need to worry about the first half of this expression, which is manageable. It is evident that if $T_a \leq t$, then at some point from 0 to t the Brownian motion process hit a, and by the stationary increments assumption, once it moves away from a it behaves just as it would at $t = 0$. Since this random walk is always symmetric, there is then an equal probability that the process will be above or below a at time t. Therefore, $P\{X(t) \geq a \mid T_a \leq t\}$ equals one half. Our equation is now simplified greatly, and we see that

$$P\{T_a \leq t\} = 2P\{X(t) \geq a\}$$

We know that $X(t)$ is distributed normally, however, so we can convert the right-hand side to the probability density function for a normal distribution. Therefore we have

$$P\{T_a \leq t\} = \frac{2}{\sqrt{2\pi}\sqrt{t}} \int_a^{\infty} e^{-x^2/2t} dx$$

Using the strategy of change of variables, we set $y = x/\sqrt{t}$. This leaves us with $dx = \sqrt{t}\, dy$, and this simplifies the expression to

$$P\{T_a \leq t\} = \frac{2}{\sqrt{2\pi}} \int_{a/\sqrt{t}}^{\infty} e^{-y^2/2} dy$$

This expression holds for all $a > 0$. But what about for $a < 0$? Luckily, the symmetry of the Brownian motion process allows us to find $P\{T_a \leq t\}$without any more derivation. We can just assume that the process will behave the same if it is on either side of zero, so this leads to

$$P\{T_a \leq t\} = \frac{2}{\sqrt{2\pi}} \int_{|a|/\sqrt{t}}^{\infty} e^{-y^2/2} dy$$

for all a, where $|a|$ is the absolute value of a (Ross 455-456). Notice that this expression is a definite integral, which means that it can be evaluated at the given bounds. This integral is very difficult to evaluate in practice, however, since it is not an ordinary exponential integral. Fortunately for us, though, we are interested in the probability density function of T_a, which is the derivative of the above cumulative density with respect to t. Carrying out this calculation, we obtain

$$f_{T_a}(t) = \frac{xt^{-3/2}}{\sqrt{2\pi}} e^{-x^2/(2t)}$$

Thus, this is the precise expression for the probability of a Brownian motion process *X*(*t*) hitting a value *a* by a certain time *t* (Taylor 494). Another aspect of Brownian motion that is closely related to hitting times is the maximum value attained by the process over the interval [0, *t*]. By continuity, we see that

$$P\{ \max_{0 \le s \le t} X(s) \ge a \} = P\{T_a \le t\}$$

because if the maximum values *X*(*s*) is greater than *a*, that implies that the hitting time of the process is less than *t*, and vice-versa. Therefore these two events are described by the same distribution (Ross 456).

Earlier, the Gambler's Ruin problem was discussed in much detail, but it turns out that there is more to be said about this important problem, at least in relating it to Brownian motion. It has been said many times that Brownian motion arises from a limit of symmetric random walks, which are types of Markov chains, and this property will be used to tie Brownian motion together with the Gambler's Ruin problem. Assume that we want to know the probability of a Brownian motion process *X*(*t*) going up by *A* units before it drops *B* units below its original starting point. Earlier we denoted *N* as the goal of the gambler, but what it really represents is the total number of steps in the random walk, which for this problem equals $(A+B)/\Delta x$, where Δx is the length of a step in the process (in the earlier problem, *A* equaled the number of units away from the goal *N* the gambler was, *B* was equal to the number of units the gambler possessed, and Δx was equal to 1). This implies that *i*, which is the starting number of units, is equal to $B/\Delta x$

(which reduced to simply B before). Therefore, from the equation derived before for p = 1/2, P_i equals

$$\frac{i}{N} = \frac{B/\Delta x}{(A+B)/\Delta x} = \frac{B}{A+B}$$

So, as we let Δx go to zero, it is obvious that for Brownian motion the probability of the process going up A units before it goes down B units is equal to $B/(A+B)$ (Ross 456).

An example of much interest is the fluctuation of stock prices. There are many different factors to consider when modeling stock prices with Brownian motion, and this will be discussed later, but roughly we can discuss the likelihood of a stock behaving in a certain way by using logarithms. If $X(t)$ represents the value of a certain stock at time t, then log $X(t)$ is approximated by Brownian motion, which is useful. This is often called geometric Brownian motion (Taylor 514). Applying what was stated above, it is evident that the probability of a stock whose current price is x attaining the value of αx before dropping to x/β, where α and β are constants greater than one, is equal to

$$\frac{\log \beta}{\log \alpha + \log \beta}$$

This closely resembles the expression derived above, and makes sense considering how we defined this special form of Brownian motion (Ross 457).

A Bit More

Reflected Brownian Motion

Much has been said up to this point about the Brownian motion stochastic process, but most real-world applications have certain constraints that force us to modify how we look at this mathematical process. For example, consider the situation that led to the discovery of Brownian motion: pollen grains moving in a fluid medium. One would think that these pollen grains would not able to move an infinite distance in any direction, since they cannot travel outside the fluid they are moving in, so we must consider a new take on Brownian motion that accounts for such boundaries. Consider once again $X(t)$, which is a standard Brownian motion process. Then the new process $R(t)$ is equal to

$$|X(t)| = \;\; X(t), \qquad \text{if } X(t) \geq 0$$
$$|X(t)| = -X(t), \qquad \text{if } X(t) < 0$$

$R(t)$ is known as reflected Brownian motion (at the origin). Reflected Brownian motion can never take on negative values, as is evident from its definition, so this means that the process "bounces" the motion off of a boundary at $X(t) = 0$. This is why this process more accurately models the motion of some particle contained within some medium, because in reality when the particle strikes the boundary of the medium it rebounds off of the boundary instead of passing through it.

Finding the mean and variance of this new process is relatively easy, because the moments of $R(t)$ and $|X(t)|$ are the same. Taking the expectation of $|X(t)|$, we see that

$$E[\,|X(t)|\,] = E[R(t)] = \int_{-\infty}^{\infty} |x| \phi_t(x)dx$$

where $\phi_t(x)$ is the probability density function for a normal distribution with mean zero and variance t. In other words, it is the density function for the Brownian motion process. As a reminder, the density function for a normal distribution is

$$f_Y(y) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{1}{2}(\frac{y-\mu}{\sigma})^2}$$

defined for all y (Larsen 307). From this, we see that the above expression becomes

$$2\int_0^\infty \frac{x}{\sqrt{2\pi t}} e^{-\frac{x^2}{2t}} dx$$

because $R(t)$ from negative infinity to zero is identical to $R(t)$ from zero to positive infinity. We can simplify the integration by just worrying about the positive side of the integration and multiplying by two. This integral is very similar to the integral dealt with earlier, and it is even solved using the same substitution, $y = x/\sqrt{t}$. Integrating and evaluating at the bounds leaves us with

$$E[R(t)] = \sqrt{2t/\pi}$$

We can also calculate the variance easily using

$$\begin{aligned} Var[R(t)] &= E[R(t)^2] - \{E[R(t)]\}^2 \\ &= E[X(t)^2] - 2t/\pi \\ &= t - 2t/\pi \\ &= (1 - \frac{2}{\pi})t \end{aligned}$$

Here $E[X(t)^2]$ is calculated in the same manner as the mean of $R(t)$ was calculated above, evaluating a similar integral (Taylor 498-499).

<u>Absorbed Brownian Motion</u>

Similar to reflected Brownian motion, absorbed Brownian motion deals with the behavior of Brownian motion at $x = 0$, except as the name implies, absorbed Brownian motion deals with processes that stay at zero once they reach this level. More precisely, consider a time τ at which a Brownian motion process first reaches zero. Then we define absorbed Brownian motion as

$$\begin{aligned} A(t) &= X(t), \quad &\text{if } t \leq \tau \\ A(t) &= 0 \quad &\text{if } t > \tau \end{aligned}$$

This type of Brownian motion process is useful when trying to model the price of a stock that may become bankrupt at some point in the future. We want to be able to obtain an idea of how such a process might behave at a future time, so in order to do this we consider the transition probability, which is the probability of a process in one state moving to another particular state (Taylor 95). Assuming that $x > 0$ and $y > 0$, we have

$$G_t(x, y) = P[A(t) > y \mid A(0) = x]$$
$$= P[X(t) > y,\ \min_{0 \le u \le t} X(u) > 0 \mid X(0) = x]$$

where u is a transient time in the interval $[0, t]$. We are able to relate the two separate probabilities in the second expression since if the minimum of $X(u)$ attained on $[0, t]$ is equal to zero, by definition of absorbed Brownian motion $X(t)$ will equal zero for the remainder of the process. In order to learn more about $G_t(x, y)$, we consider the broader case of which it is a part

$$P[X(t) > y \mid X(0) = x] = G_t(x, y) + P[X(t) > y,\ \min_{0 \le u \le t} X(u) \le 0 \mid X(0) = x]$$

Here the expression on the left is broken up into the two separate cases, $\min_{0 \le u \le t} X(u) > 0$ and $\min_{0 \le u \le t} X(u) \le 0$. One would think that the latter expression would lead to a zero probability, since $\min_{0 \le u \le t} X(u) \le 0$ implies that the process would equal zero at t. We can use the idea of reflection, however, to show that there is a path of equal likelihood that acts as the mirror image of $X(t)$ over the horizontal axis. It is this path that is relevant for the second term in the above expression (Taylor 500).

With this knowledge in hand, we claim that

$$P[X(t) > y,\ \min_{0 \le s \le t} X(u) \le 0 \mid X(0) = x]$$

$$= P[X(t) < -y,\ \min_{0 \le u \le t} X(u) \le 0 \mid X(0) = x]$$

$$= P[X(t) < -y \mid X(0) = x] = \Phi_t(-y - x)$$

where Φ_t is the cumulative density function of ϕ_t, the probability density function defined earlier. The inequality $X(t) < -y$ is relevant here since a reflection of a path that satisfies $X(t) > y$ will satisfy $X(t) < -y$. Also, we are able to drop the second condition because if we are looking for $X(t) < -y$, then it is unnecessary to require that $\min_{0 \le u \le t} X(u) \le 0$. Inserting this new expression into the original equation for $P[X(t) > y \mid X(0) = x]$, we get after moving some terms around

$$G_t(x, y) = P[X(t) > y \mid X(0) = x] - P[X(t) < -y \mid X(0) = x]$$

$$= 1 - \Phi_t(y - x) - \Phi_t(-y - x)$$

$\Phi_t(-y - x)$ is equal to $\Phi_t[-(x + y)]$, however, so we can write $1 - \Phi_t(-y - x)$ as $\Phi_t(y + x)$.

$$G_t(x, y) = \Phi_t(y + x) - \Phi_t(y - x)$$

$$= \int_{y-x}^{y+x} \phi_t(z)dz = \Phi(\frac{y+x}{\sqrt{t}}) - \Phi(\frac{y-x}{\sqrt{t}}) \quad \text{(Taylor 501)}$$

Now that most of the basic manifestations of Brownian motion have been investigated, it is reasonable to move on to applications.

III

Applications of Brownian Motion

The most prominent applications of Brownian motion used today involve its relevance to stock prices and stock options. These are not the only applications of Brownian motion, however, and others will be discussed, although in a more cursory manner. Before we can dive into applications of Brownian motion at all, though, there is a bit more that needs to be said about this process.

Subtlety Within Insanity

Most real-world Brownian motion processes exhibit the property that over time the mean of the process slowly (and constantly) shifts either upwards or downwards. This property is known as the drift parameter, and it is represented by μ. Recall that the variance of Brownian motion can also vary, and the parameter that measures this value is c. Up until this point, we have assumed that $c = 1$, but now we will allow c to take on other values with the intention of witnessing different types of Brownian motion. Taking these constants into consideration, we have $Y(t)$, our new Brownian motion process, equal to

$$Y(t) = \mu t + cX(t)$$

where $X(t)$ is a standard Brownian motion process as defined earlier and $t \geq 0$. When looking at the increments of this process, namely $Y(t + s) - Y(t)$, we know from before that they have normal distributions with mean μs and variance $c^2 s$. Assuming $Y(0) = x$, we can get a conditional density for this process:

$$P[\text{Y}(t) \le y \mid X(0) = x] = P[\mu t + cX(t) \le y \mid cX(0) = x]$$

$$= P[X(t) \le \frac{y - \mu t}{c} \mid X(0) = \frac{x}{c}]$$

$$= \Phi_t(\frac{y - x - \mu t}{c}) = \Phi(\frac{y - x - \mu t}{c\sqrt{t}}) \quad \text{(Taylor 508)}$$

We now return once again to the important Gambler's Ruin Problem, but this time another twist is added in. The goal of the problem is the same as before, except now the idea of drift and variance will be included, complicating the process even further. Define T_{ab} to be the time at which the Brownian motion process exits the interval [a, b], or more precisely,

$$T_{ab} = \min\ [t \ge 0;\ Y(t) = a \text{ or } Y(t) = b]$$

As before, we are interested in the probability of T_{ab} bringing the process to the high side of the interval, namely b. According to Taylor and Karlin, this probability is

$$P[Y(T_{ab}) = b \mid Y(0) = x] = \frac{e^{-2x\mu/c^2} - e^{-2a\mu/c^2}}{e^{-2b\mu/c^2} - e^{-2a\mu/c^2}}$$

(For a proof of this result, see Taylor page 509). This equation allows us to predict the behavior of different real world processes, but as was said before, the most prominent application of this type of mathematics is the forecasting of stock prices. An example will help illustrate how this equation works.

Suppose that share prices of Duncan Motors (DUNC) follow a Brownian motion model with drift. Here the drift, μ, can be thought of as the long term growth of the company, and c can be thought of as a measure of the short-term fluctuations of the stock. The higher c is, the more severe the fluctuations are. Also, let the time t represent the number of weeks of trading that have taken place, where t reflects continuous time. In

2007, Duncan Motors is expecting growth of ten percent, and on any given week of trading, the fluctuation of the stock is given by $c^2 = 5$. Suppose a shareholder buys ten shares of Duncan Motors at a price of \$50, and plans on selling them if the price increases to \$80 or drops to \$35. What is the probability that the shareholder sells his shares at a profit?

To solve this, we need to convert the company's growth to a weekly growth, since that is the unit of time being dealt with in this situation. Therefore, we have

$$\mu = .10/52 = .002 \qquad \text{and}$$

$$2\,\mu/c^2 = 2(.002)/5 = .0008$$

Plugging this into the equation with $x = 50$, $a = 35$ and $b = 80$, we see that

$$P(\text{profit}) = \frac{e^{-(50)(.0008)} - e^{-(35)(.0008)}}{e^{-(80)(.0008)} - e^{-(35)(.0008)}} = .337$$

It seems that our investor might want to rethink his investment strategy. What would happen if the shareholder decided to sell sooner? Suppose he decides to sell at \$55, a nice ten percent profit. One would expect the probability of him selling at a profit to increase. We see that

$$P(\text{profit}) = \frac{e^{-(50)(.0008)} - e^{-(35)(.0008)}}{e^{-(55)(.0008)} - e^{-(35)(.0008)}} = .750$$

which is much more favorable to the investor. This example shows that even when a Brownian motion process is drifting upward, the short-term fluctuations can affect the process enough to temporarily nullify this growth.

The Logic of Logs

We briefly mentioned earlier how it is often useful to use logarithms to model different stochastic processes. It is now time to explore this idea in more depth, with the hopes of understanding these types of processes more thoroughly. By definition, a geometric Brownian motion process $Z(t)$ is a stochastic process such that $W(t) = \log Z(t)$ is a Brownian motion process with variance parameter c^2 and drift parameter $\mu = \alpha - \frac{1}{2}c^2$, where α is the drift parameter of $Z(t)$. Using this idea we can write any geometric Brownian motion process $Z(t)$ with initial value $Z(0) = z$ as a function of a standard Brownian motion process $X(t)$ in the following way:

$$Z(t) = ze^{W(t)} = ze^{(\alpha - 1/2c^2)t + cX(t)}$$

This new model is appealing to many for a few reasons. Most prominently, $Z(t)$ can never be negative, which is important if one wants to model the behavior of a stock or other market entity. In addition, $Z(t)$ follows a long term exponential decay or growth trajectory, thanks to the presence of e, and this also more accurately describes many situations in trading. One other property that $Z(t)$ exhibits, much like standard Brownian motion, is the independent increments assumption. The expression of intervals for geometric Brownian motion, however, takes a different form thanks to the presence of exponential quantities. For times $t_0 < t_1 < \ldots < t_n$, the ratios (increments)

$$\frac{Z(t_1)}{Z(t_0)}, \frac{Z(t_2)}{Z(t_1)}, \ldots, \frac{Z(t_n)}{Z(t_{n-1})}$$

are all independent of one another. This means that the relative percentages of each consecutive pair of random variables do not depend on one another, which is useful when trying to understand the incremental behavior of the process.

Much like with reflected and absorbed Brownian motion, it is useful for us to find the expectation and variance of geometric Brownian motion in order to better understand what is happening when this process runs. Before working with the actual function *Z*(*t*), it is necessary to consider the area under a normal curve, in order to reveal an important fact that will be of use later. Assuming that we are working with a normally distributed variable *u* with unknown parameters, we represent the area under the normal curve by

$$1 = \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(u-\lambda)^2} du$$

where λ is a constant. Simplifying the integrand, we see that this equals

$$\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(u^2-2\lambda u+\lambda^2)} du$$

$$= e^{-\frac{1}{2}\lambda^2} \int_{-\infty}^{\infty} e^{\lambda u} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}u^2} du$$

$$= e^{-\frac{1}{2}\lambda^2} E[e^{\lambda\xi}]$$

where ξ is a standard normally distributed variable, meaning that its parent distribution has mean zero and variance one. This switch is made because calculating the mean of *Z*(*t*) involves calculating this expression. Doing the simple algebra leaves us with

$$E[e^{\lambda\xi}] = e^{\frac{1}{2}\lambda^2}$$

For this specific case, $\xi = X(t)/\sqrt{t}$ since dividing by $\sqrt{t}$ normalizes the variance. Calculating the mean of *Z*(*t*) is now simple:

$$E[Z(t) \mid Z(0) = z] = zE[e^{(\alpha-\frac{1}{2}c^2)t+cX(t)}]$$

$$= ze^{(\alpha-\frac{1}{2}c^2)t} E[e^{c\sqrt{t}\xi}]$$

$$= ze^{(\alpha - \frac{1}{2}c^2)t} \; e^{\frac{1}{2}c^2 t}$$

$$= ze^{\alpha t}$$

There is an important thing that needs to be mentioned concerning the mean of $Z(t)$ and other stochastic processes in general. When α is positive less than half c^2, the expression $\alpha - \frac{1}{2}c^2$ is less than zero, which implies that $W(t)$, the Brownian motion process with drift $\alpha - \frac{1}{2}c^2$, is drifting toward zero. The Law of Large Numbers says that after a certain amount of time, the limit of this process must eventually approach negative infinity, since the drift is negative (Stark). But $Z(t) = ze^{X(t)}$, so this means that as $W(t)$ goes to negative infinity, $Z(t)$ must approach zero. Putting this all together, we have a process whose expectation is increasing, yet at the same time the limit of the process is approaching zero. This is a perfect example of how it is necessary to consider as many facets of stochastic processes as possible when attempting to understand the behavior of the process, because sometimes looking at only one dimension hides critical information.

The same method as was used above also applies to calculating the variance of $Z(t)$. We already have $E[Z(t)]$, so now all we need is $E[Z(t)^2]$ in order to use the formula

$$\text{Var}[Z(t)] = E[Z(t)^2] - E[Z(t)]^2$$

$$E[Z(t)^2 \mid Z(0) = z] = z^2 E[e^{2W(t)}]$$

$$= z^2 E[e^{2(\alpha - \frac{1}{2}c^2)t + 2cX(t)}]$$

$$= z^2 e^{2(\alpha + \frac{1}{2}c^2)t}$$

Once again the change in variables to $\xi = X(t)/\sqrt{t}$ and appealing to result derived earlier gives us the result in the last line of the above calculation. We can now use

$$\begin{aligned}\text{Var}[Z(t)] &= E[Z(t)^2] - E[Z(t)]^2 \\ &= z^2 e^{2(\alpha+\frac{1}{2}c^2)t} - z^2 e^{2\alpha t} \\ &= z^2 e^{2\alpha t}(e^{c^2 t} - 1) \qquad \text{(Taylor 515-516)}\end{aligned}$$

In keeping with the recurring example of this work, we will now turn again to the Gambler's Ruin Problem to illustrate some of the essential components of geometric Brownian motion. Recall before that the intervals of geometric Brownian motion are represented by ratios of points rather than subtractions, so the relevant probability of the Gambler's Ruin Problem stated before becomes

$$T_{ab} = \min\left\{t \geq 0; \frac{Z(t)}{Z(0)} = A, \frac{Z(t)}{Z(0)} = B\right\}$$

where the interval $[A, B]$ is defined such that $A < 1$ and $B > 1$. This definition of A and B is a nod to the exponential nature of this type of problem, and the values are scaled with respect to the starting point of the process. A variation of the theorem posited in the previous section applies to this situation as well, and once again we are interested in the probability of the process exiting the interval at B and not A. Assuming geometric Brownian motion with variance c^2 and interval endpoints $A < 1 < B$, it is true that

$$P\left\{\frac{Z(t)}{Z(0)} = B\right\} = \frac{1 - A^{1-2\alpha/c^2}}{B^{1-2\alpha/c^2} - A^{1-2\alpha/c^2}}$$

Assume that once again we are interested in our investor's stake in Duncan Motors, except this time we will analyze the fluctuations of the stock using geometric Brownian motion. From before, we know that $2\alpha/c^2 = .0008$, therefore $1 - 2\alpha/c^2 = .9992$. How do we define *A* and *B* though? We need to scale each action price of the stock by the starting price, \$50, so to do this we just divide the two values by fifty. This gives us (for the first variation of the problem) $A = .7$ and $B = 1.6$, and using the formula we see that

$$P(\text{profit}) = \frac{1 - .7^{(.9992)}}{1.6^{(.9992)} - .7^{(.9992)}} = .333$$

which is not too far from the value obtained by using ordinary Brownian motion with drift. What happens when we consider the investor's revised strategy, though? Changing our *B* to 55/50 = 1.1,

$$P(\text{profit}) = \frac{1 - .7^{(.9992)}}{1.1^{(.9992)} - .7^{(.9992)}} = .750$$

which is identical to what we obtained using the non-geometric model. It may not seem that this model is any better than what was used before, but it is more accurate when dealing with prices that are closer to the starting price, and also when α takes on larger values than the modest growth detailed here. In other words, geometric Brownian motion is more accurate when trying to predict the short term behavior of a more dynamic process (Taylor 516).

Three Men Turn the Financial World Upside-Down

Since the most recent topic discussed was geometric Brownian motion, it makes sense to begin with the application that ties directly in with this process, namely the Black-Scholes option pricing formula. To provide the motivation for this example, consider again our investor who is closely following the price of stock in Duncan Motors. He sees that the current price of the stock is \$74, and he believes that the price of the stock will increase in the coming year, due to positive earnings reports reported by the company's young CEO. Instead of buying shares of stock at $74 apiece, the investor has the opportunity to purchase what is known as a "call" option, which gives the investor the opportunity to buy shares of Duncan Motors at the fixed price of $74 at any time during a predetermined interval, no matter how much the stock's price increases. These call options are offered by brokers at a fraction of the price of the actual stock, too, so it is an enticing method of investing for many people.

For example, suppose that a call option for Duncan Motors is being sold for $8 per share at a striking price of $75, which means that if the price of the stock were to move above $75, the investor would then have the chance at any time to exercise the option, purchasing the stock at the price of $75. So if the investor were to wait until the price of the stock reached $90, he could exercise his option, purchasing the stock at $75, then immediately sell at $90 for a $7 profit per share, when you factor in the price of the option. It is a low risk strategy for the investor, since he is only in danger of losing the $8 option fee if the stock price ends up dropping. The seller of the option takes on the most risk in the transaction, because if the stock skyrockets, the seller will be obligated to sell the stock to the option holder for well below its market value, generating a huge loss on his side. This created an interesting problem that puzzled the brightest minds in the financial world for many years, and many people wondered what the appropriate price was for these call options.

It was not until 1973 when Fisher Black, a financial consultant, Myron Scholes, a finance professor at MIT, and Robert Merton, another MIT instructor who was yet to earn his doctorate, created a new analysis of the problem using geometric Brownian motion as its foundation (Bernstein 311). Many other mathematicians and financial analysts had tried working with Brownian motion before, but Black, Scholes and Merton succeeded in their analysis by ignoring a previous assumption that the option should have a higher average return than the actual stock because the option writer (the seller of the option) was in a position of unlimited risk, since theoretically the price of the stock could go to infinity. Black, Scholes and Merton circumvented this idea with an approach known as "program trading," where the option writer both buys and sells the stock in question concurrently in a manner that matches the return of the option, thus eliminating any randomness in the process of trading stocks. This new strategy also bears a couple of new requirements. First, the call option must have the same rate of return as other investments that are risk-free (such as certificates of deposit or U.S. Treasury Bonds), otherwise the writer could use the strategy to discover investing opportunities that bear no risk yet yield extremely high profits. Second, this new strategy requires that the holder of an option not exercise it until the option expires, regardless of when the striking price is eclipsed, since holding an option is a low-risk undertaking. These two assumptions play an important role in the work of these three brilliant men, and this revolutionary way of looking at options trading helped lead to the formulation of an option pricing formula that is still widely used today.

Once the Black-Scholes (Merton was not fortunate enough to get his name on the final product) option pricing formula became known, it was adopted by financial institutions quickly, despite its mathematical complexity and its pertinence to only an ideal financial realm. The entire derivation will not be discussed here, as it would occupy several pages alone, and only the result is of any interest. To start, let $S(t)$ be a geometric Brownian motion process with drift α and variance c^2, where $S(t)$ represents the price of a given stock at some time t. Also, let $F(z, \tau)$ represent the price of the corresponding call option, where z is the current price of the stock and τ is the time until the option expires. So, the option price of a stock trading at \$30 for one year would be represented by $F(30,1)$, since t is usually defined in years. In addition, let a be the striking price, which again is the price at which the holder can exercise the option. Following the guidelines of Black and Scholes, the time to decide whether or not to exercise the option comes at $\tau = 0$, or the end of the option period. If at this time $z > a$, the holder will exercise the option for a profit $z - a - F(z_0, \tau_0)$ per share, where z_0 and τ_0 are the initial price of the stock and the duration of the option, respectively. On the other hand, if $z \leq a$, then the holder will not exercise the option and will lose $F(z_0, \tau_0)$ for each option purchased. This gives us a simple way to express the end result of any particular call option:

$$F(z, 0) = (z - a)^+ = \max\{ z - a, 0\}$$

Combining this equation with the assumptions made by Black and Scholes discussed above, we see that

$$F(z, \tau) = e^{-rt} E[(Z(\tau) - a)^+ \,|\, Z(0) = z]$$

where r is the rate of return for risk-free investments as defined earlier, and $Z(t)$ is another geometric Brownian motion process with drift r and variance c^2. This equation makes sense because it is defined in terms of the state of the process at the expiration time of the option (τ) and it factors in risk-free investments, remaining consistent with the theory of Black and Scholes.

The derivation of their valuation formula begins by defining $Z(t)$ by

$$Z(\tau) = ze^{(r - \frac{1}{2}c^2)\tau + c\sqrt{\tau}\xi}$$

where $\xi = X(t)/\sqrt{\tau}$, a standard Brownian motion process divided by the square root of the duration of the option. This change of variables was used earlier, so it comes as no surprise that it is useful in this situation as well. The proof continues by establishing a condition v_0 for which a is a lower bound of $Z(\tau)$, then using the initial formula stated above to evaluate $e^{r\tau} F(z,\tau)$, then by extension $F(z, \tau)$. The actual derivation is much more involved than this brief sketch, but it turns out that the Black-Scholes option pricing formula becomes

$$F(z, \tau) = z\Phi\left(\frac{\log(z/a) + (r + \frac{1}{2}c^2)\tau}{c\sqrt{\tau}}\right) - ae^{-r\tau}\Phi\left(\frac{\log(z/a) + (r + \frac{1}{2}c^2)\tau}{c\sqrt{\tau}}\right)$$

where: z = current price of the stock

a = striking price of the option

τ = time to expiration of the option

r = rate of return of risk-free investments

c = volatility of the stock

At first glance, this equation seems very formidable, but in reality it is not too difficult to work with, since four of the five parameters listed above are simple to find or calculate, and the cumulative distribution for a standard normal distribution, Φ, is always readily available. The only difficulty presented by this equation is the requirement that the volatility of the stock, c (also the standard deviation), be known. Various methods of parameter estimation can be used to approximate c, but these methods all base their estimates off of previous data, whereas it is the future volatility that affects the appropriate price of a call option.

An interesting method used by some investors is to work with the Black-Scholes pricing formula, plugging in the known values of the four parameters *and* the current value of the option in order to solve for the current volatility of the stock. Once again, this does not address the problem of knowing the future volatility of the stock, but it helps give investors a better idea as to whether or not they want to work with options being offered at a certain price. This type of estimated parameter is known as implied volatility (Taylor 518-520). One drawback to this method is that its precision is questionable at best. One could proceed through the necessary calculations several times for identical options at different striking prices and get a different volatility each time, which should not be the case. One can plot the different values of implied volatility against the striking price of the option in question and obtain a plot that looks like this (Hafner 39):

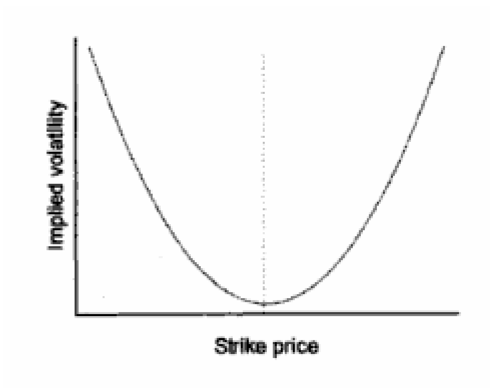

Figure 1: Illustration of a Volatility "Smile"

Because of such behavior, this type of relation is known as a volatility "smile" (Roman 271).

Below is a table that compares theoretical Black-Scholes option prices with actual prices of options in IBM stock on February 26, 1997. At the time IBM traded at $146.50

per share, risk-free investments were providing about a five percent annual return (r = .05) and the volatility was estimated to be c = .3 (Taylor 520-521).

Striking Price (a)	*Years to Expiration* (τ)	*Black-Scholes Price* [$F(z, \tau)$]	*Actual Price*
130	1/12 (1 month)	$17.45	$17.00
130	2/12	$18.87	$19.25
135	1/12	$13.09	$13.50
135	2/12	$14.92	$15.13
145	1/12	$6.14	$5.50
145	2/12	$8.52	$9.13
155	1/12	$2.18	$1.63
155	2/12	$4.28	$4.00

Figure 2: Theoretical and Actual Options Prices for IBM Stock: February 26, 1997

We can observe some different trends from this chart that exhibit how the Black-Scholes model works. It is obvious that the bigger the difference between the striking price and the market price of the stock, the cheaper the option will be, since the chances of the stock reaching the striking price and remaining at a level above it are much less than if the striking price were closer to the market price. We can also see that as the lifetime of the option increases, so does the price of the option. These findings are intuitive, but can be verified nonetheless by thinking about the behavior of Brownian motion. The long-term motion of the process is determined by its drift, so smaller striking prices and longer times to expiration for options give the process a better chance at naturally drifting above its target. Therefore, since the probability of turning a profit with these favorable conditions increases, so does the price of the option. Notice how the actual prices of the options are all close to the Black-Scholes estimates, which tells us that Wall Street does in fact employ valuation formula that is similar to the Black-Scholes model.

Fractals

The study of fractals is an important related topic that plays a role in some applications of Brownian motion. Fractals, which is an abbreviation of “fractional dimension,” can be thought of in a strictly graphical manner or a pure mathematical one. For the purposes of remaining close to Brownian motion, only the graphical interpretation will be discussed. In the simplest terms, a fractal is a type of image that exhibits certain types of patterns on infinitely many different scales. More precisely, fractals exhibit a property known as self-similarity, which means that if any portion of a fractal is blown up to any scale, no matter how small, the magnified image will have an identical pattern as the original image (Lee 3-4). Many fractal-like images can occur in nature, in structures such as mountains, clouds, and even broccoli! These “fractals” are not true fractals, however, but they can be treated as such in a statistical manner (Strogatz 398). Ideal fractals can be constructed using iterated functions or processes, such as the Cantor set.

The Cantor set is a simple to visualize early on the process, but as the number of iterations of the process increases its complexity increases greatly. To begin, imagine a line segment spanning the interval [0, 1], and remove its middle portion, leaving segments of identical length on each side of this new gap. Continue by repeating this for each of the two new line segments, and then again for the four new segments created thereafter. Repeating this process an infinite number of times gives the Cantor set. The Cantor set, a fractal, can be seen below (Ellis).

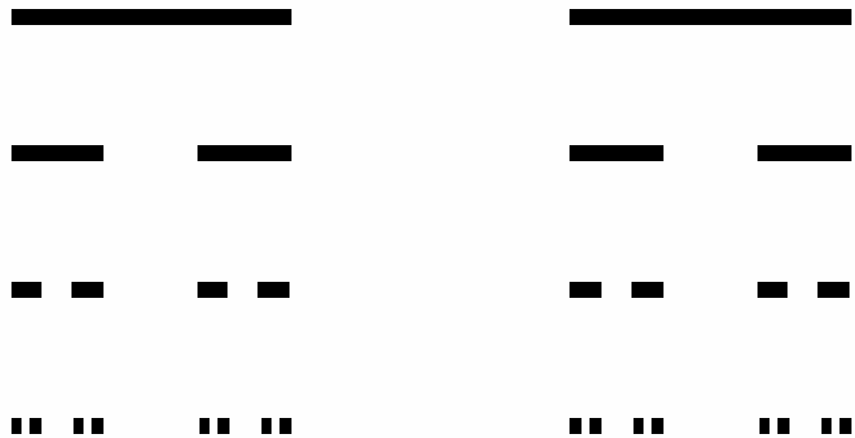

Figure 3: The Cantor Set

It is easy to see at these early stages the self-similarity of the structure at different scales, meaning that if you were to zoom in on the bottom-left section of the set, you would see a picture that is identical to the top row. In addition, the Cantor set does not have an integer dimension, which is another important property of fractals. What is meant by this is that the Cantor set is somewhere in between a point, which has dimension zero, and a line, which has dimension one (Lee 4). This may not seem intuitive, but when one realizes that the infinite Cantor set is an infinite amount of points with finite spaces in between them, this classification makes sense. The fractal dimension of the Cantor set is actually equal to .63, which makes more of a line than a point (Strogatz 402).

One may think how this at all ties into Brownian motion, but there is in fact a link. Brownian motion in fact has a fractal dimension of the number of dimensions the process is acting in, because of its nature as a random process. Consider a particle that exhibits Brownian motion in two dimensions, meaning that its motion along a surface is entirely random. In plotting the particle's trajectory to a specific point, the probability of the particle reaching that exact point is miniscule, implying that the particle would fill almost the entire surface before it actually reaches that point. This equates the path of the particle to the surface itself, therefore the Brownian motion process is a two-dimensional entity. A more significant synthesis of these two ideas is the fractional Brownian motion model, which allows us to approximate rough surfaces in nature, such as mountains and clouds (mentioned earlier), as products of Brownian motion processes (Lee 4-5). This theory leads to other interesting applications that are at the forefront of many different fields today.

Other Fun Things Involving Brownian Motion

<u>Medical Imaging</u>

Of the many applications of Brownian motion in use today, its application to medical imaging has arguably been the most successful and productive. Brownian motion (fractional Brownian motion, to be specific) is used today in two main capacities in this field, classification of tissues and other structures, and also discerning textures and edges of an image. This is definitely useful considering that the patterns shown on medical images often exhibit high fractal dimensions, which means that it is difficult to understand their patterns and complexity without the assistance of a tool such as Brownian motion. The randomness in these images stems from both the natural structure of the object being viewed (much like the fractal structure of clouds and mountains), and the noise (interference) that results from the processing of the image itself. By treating the natural randomness of the structure as Brownian motion, images of this type can be enhanced by highlighting the natural randomness to overcome the random noise found in the image.

Specifically, this method can be used to determine whether or not liver images are normal or abnormal. In the past, traditional statistical methods had been used to differentiate between these two types of images, utilizing such techniques as Fourier transforms and linear regression. Today, however, it is becoming more common for a fractional Brownian motion model to be used to represent the features of such an image by analyzing the image at different scales. By doing this, it is possible to better analyze the image because more than one fractal dimension can be applied to the image, accounting for its non-ideal fractal behavior. This method can be extended to image segmentation and edge detection. By calculating the fractal dimension of each pixel in the image and using this information to create a transformed image of the liver, one could enhance the edges of the image without increasing the random noise in the image, making it more useful than the original image.

The ideas described above could lead to a new type of procedure to detect liver problems, such as hepatitis and cirrhosis. Today, the standard procedure for testing the presence of these abnormalities is known as “needle biopsy,” a highly invasive procedure that is not effective all the time. This alternative procedure, which would require no surgery whatsoever, could end up being more effective in catching these problems and less invasive, which would provide less headaches for both doctors and patients. It appears that this type of procedure could very well become reality soon. The work of Santanu Basu, K.S. Chan and Joseph Barba looked at different types of cells, such as breast and bronchial cells, and tried to use this fractional Brownian motion method to catalog differences between healthy cells and cancerous cells. Their results showed that the range of scales on which the cancerous cells showed fractal behavior differed greatly than that of normal cells. Thus they concluded that this method could be used as an easy and quick test to identify cancerous cells in real patients, a truly significant breakthrough (Lee 5-6).

Robotics

The idea of robots being able to move around and do things that humans can do is both a fantastical yet popular idea that people harbor concerning life in a futuristic world. It turns out that fractional Brownian motion can help make this dream a reality, at least in assisting robots in moving wherever they want to. In their paper, Kenichi Arakawa and Eric Krotkov detail a procedure that models terrain within a certain radius using fractal geometry and elevation data obtained by a robot, which allows the robot to simulate and then execute a specific route, allowing it to move seemingly of its own volition (Lee 6). More precisely, the robot measures depths and elevations around it using a special instrument, then takes this data and recreates a three dimensional map of the terrain, using fractional Brownian motion to approximate the roughness of the terrain. This fascinating feat has already been done at Carnegie Mellon University, where they constructed a robot that was able to recreate a three dimensional terrain map within a radius of ten meters and an elevation of five meters. As a result, the robot was able to move well over both rocky and sandy terrain, which shows that this model is able to account for varying complexities of surrounding terrain. This robot was also able to construct a map spanning over seven hundred meters by compiling many smaller maps it had constructed earlier (Lee 7). These findings illustrate well that it may not be long before scientists are able to design robots that are indistinguishable from humans, from a functional standpoint at least.

Decision Making

The most interesting and least obvious application of Brownian motion can be found in the area of decision making. More precisely, Brownian motion can help determine optimal switching times in some economic activity that operates on some level of uncertainty. The main research into this application was done by Kjell Arne Brekke and Bernt Oksendal, and in their work they consider a multi-faceted production process in which they wish to find the most efficient starting and stopping sequence of running the process, given the price of starting, stopping and running the process (Lee 11). The aforementioned uncertainty of the economic system allowed them to treat it as Brownian motion (or an equivalent stochastic process). An example of such a situation would be the operation of a car manufacturing plant, which might shut down temporarily if prices of steel or other materials needed to build the cars reached a critically high level. This type of problem is known as an optimal switching problem, and Brekke and Oksendal proved that an optimal starting and stopping strategy exists for a problem where the price of a resource or other entity is modeled by geometric Brownian motion (similar to how we modeled stock prices earlier). Specifically, they found that for such a process, it is not favorable to shut down the process when the price of a certain resource becomes too high, but rather it is better to wait to see if the price naturally fluctuates back to a favorable level, which would not be outrageous behavior for geometric Brownian motion. It is understood that it costs money to start and stop the process, so starting and stopping the process too often could offset any profit generated by running the process at the most cost-effective times (Lee 11).

Another type of decision making process that Brownian motion can lend assistance to is one where a decision is made when a certain threshold requirement is fulfilled, studied by L. Romanow. An example of such a process would be the method of promotion of employees at a company or firm. In particular, consider the employee evaluation procedures at Duncan Motors, where each employee is periodically observed and assessed a score of between one and one hundred. Employees whose average score is above ninety (with a minimum of a year worked at the company) are automatically offered a promotion, when one is available, due to their excellent performance. Obviously, this threshold value would differ for other companies, due to varying personnel policies and frequency of openings at higher levels. Such a model utilizes Brownian motion as well as statistical sampling methods. The continuous state space and continuous time nature of the cumulative performance of a certain employee or group of employees fits well with a Brownian motion model, and naturally anytime data is collected and averaged the presence of statistical methods will be required. Based on such data collected, employers can make accurate decisions which affect the careers of their employees, under the assumption that observation of employees is continuous and that the only mobility within a company is upwards, which is not too much of a stretch (Lee 11-12). Thus, this model stands as another useful application of Brownian motion.

Final Thoughts

There are many other applications of Brownian motion, but it would be beyond the scope of this work to touch upon all of them. Represented here are a few of the more interesting ones being utilized today, but this does not mean that the others are not worth the attention of the reader. Brownian motion is truly a wonderful topic, because it ties in so well with the randomness of the world we live in. As humans, we always feel the need to be in control of every aspect of our lives, but every once in a while we encounter a situation where the outcome is out of our control. We are helpless in determining the outcome of such an event, but nonetheless we want to know what the result will be. This is why the studies of probability, statistics, and stochastic processes are some of the most important developments in the history of mathematics and academic thought in general. Without the tools of these disciplines at our disposal, we would be at the mercy of a random world, unable to achieve a fraction of the things we are able to do in today's world.

Brownian motion is only one topic out of many that comprises this rich sector of mathematics, but it is an important one. As this topic is studied more extensively, new applications and extensions will emerge, and this is what makes it an exciting topic to study. It is hard to imagine that men like Brown, Einstein and Weiner could envision the impact they would have on modern mathematics (especially Brown), but nonetheless it was these men who laid the foundation for further study of this topic. Thanks to this foundation, the brilliant minds of today and the future will step forward and continue to develop this field into something greater. When thinking about where Brownian motion can take us, it is easy to become overwhelmed, because its potential is unlimited and unfathomable.

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