Boston College

The Graduate School of Arts and Sciences

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ESSAYS IN FINANCIAL ECONOMICS

a dissertation

by

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submitted in partial fulfillment of the requirements

for the degree of

Doctor of Philosophy

August 2009
Acknowledgements

I would like to gratefully and sincerely thank the chair of my dissertation committee, Zhijie Xiao, along with Christopher Baum, Arthur Lewbel and Fabio Schiantarelli who also have served on the committee. I could never have reached the heights or explored the depths without their guidance, encouragement and patience. Most importantly, their mentorship was paramount in providing a well-rounded experience consistent my long-term career goal. I dedicate this work to my parents and my wife!
ESSAYS IN FINANCIAL ECONOMICS

ABSTRACT

My dissertation research examines empirical issues in financial economics with a special focus on the application of quantile regression. This dissertation is composed by two self-contained papers, which center around: (1) robust estimation of conditional idiosyncratic volatility of asset returns to offer better understanding of market microstructure and asset pricing anomalies; (2) implementation of coherent risk measures in portfolio selection and financial risk management.

The first chapter analyzes the roles of idiosyncratic risk and firm-level conditional skewness in determining cross-sectional returns. It is shown that the traditional EGARCH estimates of conditional idiosyncratic volatility may bring significant finite sample estimation errors in the presence of non-Gaussianity, casting strong doubt on the positive intertemporal idiosyncratic volatility effect reported in the literature. We propose an alternative estimator for conditional idiosyncratic volatility for GARCH-type models. The proposed estimation method does not require error distribution assumptions and is robust non-Gaussian innovations. Monte carlo evidence indicates that the proposed estimator has much improved sampling performance over the EGARCH MLE in the presence of heavy-tail or skewed innovations. Our cross-section portfolio analysis demonstrates that the idiosyncratic volatility puzzle documented by Ang, Hodrick, Xiang and Zhang (2006) exists intertemporally, i.e., stocks with high conditional idiosyncratic volatility earn abnormally low returns. We solve the major piece of this puzzle by pointing out that previous empirical studies have failed to consider both idiosyncratic variance and individual conditional skewness in determining cross-sectional returns. We introduce a new concept - the "expected windfall" - as an alternative measure of conditional return skewness. After controlling for these...
two additional factors, cross-sectional regression tests identify a positive relationship between conditional idiosyncratic volatility and expected returns for over 99% of the total market capitalization of the NYSE, NASDAQ, and AMEX stock exchanges. The second chapter examines portfolio allocation decision for investors with general pessimistic preferences (GPP) regarding downside risk aversion and out-performing benchmark returns. I show that the expected utility of pessimistic investors can be robustly estimated within a quantile regression framework without assuming asset return distributions. The asymptotic properties of the optimal portfolio weights are derived. Empirically, this method is introduced to construct the optimal fund of CSFB/Tremont hedge-fund indices. Both the in-sample and out-of-sample back-testing results confirm that the optimal mean-GPP portfolio outperforms the mean-variance and mean-conditional VaR portfolios.

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Degree: Doctor of Philosophy
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Chapter 1

Idiosyncratic Volatility, Expected Windfall, and the Cross-Section of Stock Returns

Abstract: Existing empirical studies report conflicting results on the effect of idiosyncratic risk in determining the cross-section of stock returns. This paper analyzes the roles of idiosyncratic risk and firm-level conditional skewness in determining cross-sectional returns. It is shown that the traditional EGARCH estimates of conditional idiosyncratic volatility may bring significant finite sample estimation errors in the presence of non-Gaussianity, casting strong doubt on the positive intertemporal idiosyncratic volatility effect reported in the literature. We propose an alternative estimator for conditional idiosyncratic volatility for GARCH-type models. The proposed estimation method does not require error distribution assumptions and is robust non-Gaussian innovations. Monte carlo evidence indicates that the proposed estimator has much improved sampling performance over the EGARCH MLE in the presence of
heavy-tail or skewed innovations. Our cross-section portfolio analysis demonstrates
that the idiosyncratic volatility puzzle documented by Ang, Hodrick, Xiang and Zhang
(2006) exists intertemporally, i.e., stocks with high conditional idiosyncratic volatility
earn abnormally low returns. We solve the major piece of this puzzle by pointing out
that previous empirical studies have failed to consider both idiosyncratic variance and
individual conditional skewness in determining cross-sectional returns. We introduce
a new concept - the "expected windfall" - as an alternative measure of conditional
return skewness. After controlling for these two additional factors, cross-sectional re-
gression tests identify a positive relationship between conditional idiosyncratic volatil-
ity and expected returns for over 99% of the total market capitalization of the NYSE,
NASDAQ, and AMEX stock exchanges.

*JEL Classification:* G10, C01, C02, G32

*Keywords:* idiosyncratic volatility, conditional skewness, robust estimation,
quantile regression
1.1 Introduction

The asset pricing theory of incomplete markets predicts a positive idiosyncratic volatility effect - investors should command higher expected return for bearing higher idiosyncratic risks. In the presence of information cost, Merton (1987) provides a theoretical framework where firms with high idiosyncratic volatility require high expected returns to compensate investors for holding imperfectly diversified portfolios. Some behavioral models like Barberis and Huang (2001) also predict that stocks with higher idiosyncratic volatility should earn higher expected returns. In addition, the role of idiosyncratic risk in determining stock returns has gained further importance given the evidence of Campbell, Lettau, Malkiel and Xu (2001) who find that there had been noticeable increases in both firm level volatility and the number of stocks needed to achieve a specific level of diversification over time.

However, empirical studies have yielded mixed results on idiosyncratic volatility and there has been a lively debate on the role of idiosyncratic risk in determining cross-sectional stock returns. Ang, Hodrick, Xing, and Zhang (AHXZ, 2006) find that stocks with high idiosyncratic volatility in the current month subsequently earn abnormally low returns the following month. More specifically, AHXZ find a $-1.06\%$ difference in raw average monthly returns between the two extreme quintile stock portfolios with the highest/lowest idiosyncratic volatilities. Moreover, AHXZ (2007) show that this negative idiosyncratic volatility effect is robust after controlling for other risk factors and occurs internationally across 23 developed markets. Since this negative idiosyncratic volatility effect is inconsistent with most asset pricing theories, AHXZ (2006, 2007) findings are referred to as an "idiosyncratic volatility puzzle", and have attracted much attention.

Fu (2007), Spiegel and Wang (2005), and others criticize the use of an OLS
model in the estimation of idiosyncratic volatility since it may not be able to capture the leverage effect and the time variation in the idiosyncratic volatility process. They further argue that the relationship between risk and returns should be studied intertemporally. As AHXZ (2006, 2007) primarily focus on the relationship between realized idiosyncratic volatilities and future returns, Fu (2007), on the other hand, stresses that the negative idiosyncratic volatility effect documented by AHXZ (2006, 2007) may not imply the same intertemporal effect involving conditional idiosyncratic volatility.

Some studies present strong empirical evidence of a positive intertemporal relationship between conditional idiosyncratic volatility and expected stock returns. Fu (2007) models the firm level idiosyncratic volatility with an EGARCH specification and presents a positive relationship between conditional idiosyncratic volatility and cross-sectional expected returns. Similarly, Spiegel and Wang (2006) sort stocks into 10 portfolios, based on the forecasted idiosyncratic volatility from an EGARCH model and find the exact opposite effect from AHXZ, namely stocks with high idiosyncratic volatility earn high average returns. More specifically, they show that the difference in raw average monthly returns between the decile portfolio that contains stocks with the highest idiosyncratic volatilities and the decile portfolio that contains the lowest idiosyncratic volatility stocks is 1.33%.

Given the contradictory evidence of the idiosyncratic risk effect on cross-sectional returns, the purpose of the present study is to investigate the relationship between conditional idiosyncratic volatility and cross-sectional expected stock returns in an effort to shed light on the lively debate about the idiosyncratic volatility effect.

First, we argue that the Gaussian assumption on which the traditional EGARCH estimation based is highly unrealistic at the level of the individual stock. Our empirical analysis rejects Gaussianity at the 5% significance level for over 90% of the stocks
traded on the NYSE, NASDAQ, and AMEX. Our monte carlo evidence also indicates that the traditional EGARCH estimation based on the unified (over all stocks) normality assumption may cause a severe estimation error in EGARCH estimates of conditional idiosyncratic volatility and may prove misleading in the empirical analysis. The positive idiosyncratic volatility effect reported by Fu (2007), Spiegel and Wang (2006), and Eiling (2006) disappears after relaxing the Gaussian innovation assumption on stock return processes.

Second, we propose an alternative estimator for the conditional volatility that is robust to error distributions. The proposed method does not require any distributional assumption on the stock return process. Most importantly, we allow individual stocks to have firm specific return distribution. Therefore, the present model would prove capable of capturing distinctive features among different stock return processes, including non-zero skewness and excess kurtosis. Monte carlo evidence indicates that the proposed estimator of conditional idiosyncratic volatility substantially outperform the conventional estimator in the presence of non-Gaussianity.

Third, our cross-sectional portfolio analysis confirms that the negative idiosyncratic volatility effect documented by AHXZ (2006) also exists intertemporally, or, in other words, stocks with high conditional idiosyncratic volatilities yield low average returns. In relation to the findings reported by Fu (2007) and Spiegel and Wang (2006), our results are in exact opposition. We find that a long-short trading strategy that longs stocks with the lowest conditional idiosyncratic volatility and shorts those with the highest generates an average monthly return of 1.53% from July 1964 to December 2006.

Forth, we take a further step to solve the puzzle. Our study reveals that previous studies of the idiosyncratic volatility effect fail to take into account two important factors. The first factor is the existence of a nonlinear relationship between idiosyn-
cratic risk and returns. Portfolio analysis demonstrates that the average returns for portfolios sorted on conditional idiosyncratic volatility are quite steady and slightly increases among the first few low volatility portfolios before dropping precipitously for portfolios with small stocks of excessively high levels of idiosyncratic volatility, leading to the large return difference between the two extreme portfolios.\(^1\) The second factor is that conditional skewness may play an important role in determining stock returns, especially for low price small stocks. We find that stocks with high idiosyncratic volatility are small stocks with low prices. Chen, Hong and Stein (2001) and Duffee (2002a, 2002b) show that small stocks have more positively skewed returns than large ones. Moreover, with limited obligation a low price stock is more likely to have a positively skewed return distribution than a high price stock. Therefore, a low-price small stock is more likely to exhibit lottery-like behavior by realizing an uncommon windfall profit.

According to the equilibrium model of Barberis and Huang (2007), the securities capable of delivering windfall profit may be over priced and consequently expected to earn low returns. This might help to explain why high idiosyncratic volatility stocks have low returns if those stocks are expected to realize windfall profit. In this paper, we introduce the "expected windfall" as an alternative measure for individual stock skewness. The expected windfall is an analogue to the expected shortfall while focusing exclusively on the most profitable and most unlikely returns. More specifically, for a given percentage such as 95% the expected windfall is the conditional mean of returns given the realized return among the best 5% of all possible outcomes.

After taking into account of these two additional factors, we solve the major piece of the idiosyncratic volatility puzzle by presenting strong empirical evidence

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\(^1\)The same pattern also exists in AHXZ (2006). In AHXZ (2006, Table VI B), the average return increases from 1.04\% for quintile 1 to1.2\% for quintile 3 and plunges to −0.02\% for quintile 5. This leads to a large negative difference in average returns between quintile portfolios 5 and 1.
of a positive relationship between conditional idiosyncratic volatility and expected returns using cross-section Fama-MacBeth regression tests. After controlling for the non-monotonicity in idiosyncratic volatility and individual stock conditional skewness, we discover a positive idiosyncratic volatility effect for stocks comprising 99% of the total market capitalization. This positive idiosyncratic volatility effect is consistent with the prediction of most asset pricing theories and behavioral models. While the negative idiosyncratic volatility effect concentrates among "penny-like" stocks, the most volatile, low price, micro-cap stocks with an average monthly idiosyncratic volatility of 19%, an average stock price of $7.63, an average market capitalization of $93 million and comprise less than 1% of the total market capitalization. In addition, we find that stocks expected to realize extraordinary returns by having a large expected windfall are over priced and expected to earn low returns. This finding provides strong empirical support for the theoretical model of Barberis and Huang (2007).

Finally, a cross-sectional quantile regression test further investigates the relationship between idiosyncratic risk and expected returns. While the conventional Fama-MacBeth (1973) regression test only examines predictability of cross-sectional stock returns at the average level, the proposed cross-sectional quantile regression test is able to reveal the predictive power of idiosyncratic volatility, expected windfall, and other firm characteristics on stock returns across the whole distribution of cross-sectional returns. This cross-sectional quantile regression test is intuitive and may be viewed as a more sophisticated version of portfolio analysis, in which, stocks are sorted into hundreds of portfolios based on their level of returns, and then the idiosyncratic volatility effect within each portfolio, while controlling for other factors, is pinpointed. The cross-sectional quantile regression test confirms a positive idiosyncratic volatility effect for stocks comprising 99% of the total market capital-
ization. More importantly, it shows that this positive idiosyncratic volatility effect is statistically significant.

The reminder of this paper proceeds as follows. We study estimation of conditional idiosyncratic volatility in Section 2. It is demonstrated that, for 90% stocks, the Gaussian assumption used in EGARCH estimation is rejected at the 5% significance level. A robust method to estimate conditional idiosyncratic volatility is proposed in Section 2. A monte carlo investigation reveals the advantage of the proposed method when the errors are non-Gaussian. Section 3 presents the results of portfolio analysis based on the robust estimation of idiosyncratic volatility. Section 4 performs cross-sectional regression tests to investigate the role of idiosyncratic risk in determining expected stock returns at the average level. Section 5 conducts a cross-sectional quantile regression test to further explore the idiosyncratic volatility effect. Section 6 summarizes the results and concludes.
1.2 Estimation of Conditional Idiosyncratic Volatility

Conditional volatility plays an essential role in finance. The literature on estimating conditional volatility is large. Many existing methods of volatility estimation in econometrics and finance are based on the assumption that financial returns have normal (or conditional normal) distributions. However, there is accumulating evidence that financial time series such as returns are not well approximated by Gaussian models. In particular, it is frequently found that market returns display negative skewness and excess kurtosis. Extreme realizations of returns can adversely affect the performance of estimation and inference designed for Gaussian conditions; this is particularly true of ARCH and GARCH models whose estimation of variances are very sensitive to large innovations.

In this section, we study estimation of conditional volatility and its impact on empirical analysis. We first revisit the traditional method based on Gaussian return innovations in EGARCH specifications. We investigate the Gaussian assumption and find that this assumption is rejected at the 5% significance level for over 90% of the stocks traded on major markets. Maximum likelihood estimates based this unrealistic assumption may provide poor finite sample performance and undermine the empirical results of Fu (2007) and Spiegel and Wang (2006). To accommodate the non-Gaussian feature in return distributions, we propose a more robust method based on quantile regression and minimum distance estimation.

The data in our analysis are from the CRSP database, monthly returns for stocks traded on the NYSE, AMEX and NASDAQ from July 1964 to December 2006 are collected. The monthly risk free interest rate and Fama-French three factors are
downloaded from the Kenneth R. French online data library. Firms are required to have at least 60 consecutive return observations to be eligible for estimation. The sample consists of 1,926, 356 return observations for 12,051 firms over the 510 months.

1.21 Revisiting the EGARCH Estimations for Conditional Idiosyncratic Volatility

Fu (2007) and Spiegel and Wang (2006) describe the time series of firm level idiosyncratic volatility using an EGARCH specification. More specifically, for stock $i$, the monthly return innovation $u_{i,t}$ is measured against the monthly Fama-French three factor model. The logarithm of conditional variance is specified as equation (3) and stock return innovations are assumed to be normally distributed. The EGARCH model is fitted using the maximum likelihood method with return observations for every stock. The model is

$$r_{i,t} - r_{f,t} = \alpha_i + \beta_{i,MKT}MKT_t + \beta_{i,SMB}SMB_t + \beta_{i,HML}HML_t + u_{i,t} \quad (1.1)$$

$$u_{i,t} = \sigma_{i,t} \varepsilon_{i,t}, \text{ where } \varepsilon_{i,t} \sim N(0,1) \quad (1.2)$$

$$\ln \sigma_{i,t}^2 = a_i + \sum_{l=1}^{p} b_{i,l} \ln \sigma_{i,t-l}^2 + \sum_{k=1}^{q} c_{i,k} \left[ \theta \varepsilon_{i,t-k} + \gamma \left( |\varepsilon_{i,t-k}| - \sqrt{2/\pi} \right) \right] \quad (1.3)$$

With EGARCH estimates of conditional idiosyncratic volatility, Fu (2007) and Spiegel and Wang (2006) report a positive relationship between conditional idiosyncratic volatility and cross-sectional returns. Since this positive idiosyncratic volatility effect is consistent with the prediction of asset pricing theories, the existence of the idiosyncratic volatility puzzle uncovered by AHXZ (206; 2007) is dismissed.

The unified assumption of Gaussian innovations, however, casts doubt on the finite sample accuracy of the EGARCH estimates of conditional idiosyncratic volatil-
Table 1.1: Tests for the Gaussian Innovation Assumption of the EGARCH Specification

The Gaussian innovation assumption of the EGARCH specification is tested for stocks traded on the NYSE, NASDAQ and AMEX over the time span of July 1964 to December 2006. The stocks are required to have at least 60 monthly consecutive return observations to be eligible for the estimation and tests. The second row, labeled "\( H_0: \text{Skew} = 0 \)”, refers to the percentages of stocks for which the null hypothesis of zero skewness of return innovations is rejected at the 1% and 5% significance levels, respectively. The third row, labeled "\( H_0: \text{ExKur} = 0 \)”, refers to the percentage of stocks for which the null hypothesis of no excess kurtosis of return innovations is rejected at the 1% and 5% significance levels. The last row, labeled "\( H_0: \text{Normality} \)”, refers to the overall percentage of stocks for which the null hypothesis of normality of return innovations is rejected at the 1% and 5% significance levels.

<table>
<thead>
<tr>
<th>( H_0 )</th>
<th>( \alpha = 1% )</th>
<th>( \alpha = 5% )</th>
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</thead>
<tbody>
<tr>
<td>( \text{Skew} = 0 )</td>
<td>73.8%</td>
<td>80.7%</td>
</tr>
<tr>
<td>( \text{ExKur} = 0 )</td>
<td>79.7%</td>
<td>87.5%</td>
</tr>
<tr>
<td>( \text{Normality} )</td>
<td>83.3%</td>
<td>90.3%</td>
</tr>
</tbody>
</table>

Nitri of Fu (2007) and Spiegel and Wang (2006). Such a unified normality assumption across stocks is over simplified for omitting the long-observed and well-documented high skewness and excess kurtosis in stock return processes. See for example Harvey and Siddique (1998) for analysis of US monthly stock returns. In addition, Bollerslev (1987) shows that the excess kurtosis of the stock return process is not just a result of changing volatility as it still exists after returns are normalized by their estimated conditional volatility. Gaussian based MLE can create serious finite sample estimation errors for parameter estimation and volatility prediction when the innovation distribution is different from normal.

There is a large literature on testing for normality. Two important features of non-normality are the non-zero skewness and excess kurtosis. Statistical tests for normality have been proposed based on these two important index, see, e.g. Jarque and Bera (1980) and D’Agostino, Balanger, and D’Agostino, Jr. (1990). We check the assumption of Gaussian innovations at the individual stock level based on
skewness and kurtosis. In particular, we consider tests for the following null hypotheses: $H_{01}$: skew = 0; $H_{02}$: $E x K u r = 0$; and $H_{03}$: Normality. A summarized result for individual stock returns is reported in Table 1.

As shown in Table 1, at the 1% (5%) significance level, the null hypothesis of zero skewness is rejected by 73.8% (80.7%) of stocks; and the null hypothesis of no excess kurtosis is also rejected for 79.7% (87.5%) of stocks. Overall, the normality assumption on stock return innovations is rejected at the 1% (5%) significance level for over 83.3% (90.3%) of stocks traded on the NYSE, NASDAQ and AMEX. Therefore, the Gaussian innovation assumption at the individual firm level is highly unrealistic and may introduce severe estimation error in the EGARCH estimates of conditional idiosyncratic volatility, and may mislead the empirical results of Fu (2007) and Spiegel and Wang (2006) since they assume that for every stock $i$ the time series of return innovations $\{\varepsilon_{i,t}\}_t$ is normally distributed.²

1.22 Linear TGARCH Specification

In this paper, we propose a robust volatility estimation method for a class of GARCH models based on a combination of quantile regression and minimum distance estimation. The proposed method does not impose distributional assumptions on the stock return process. In addition, we allow individual stocks to have firm specific return distributions that may be skewed with or without fat tails and may vary across stocks. The proposed method is also relatively easy to implement and has good sampling performance in our simulation experiments.

GARCH models have proven to be highly successful in modelling financial

²Not surprisingly, as reported in section 3, the positive idiosyncratic volatility effect reported by Fu(2007), Spiegel and Wang (2006) and Eiling (2006) is reversed after relaxing the Gaussian innovations assumption on stock return innovations.
data, and is arguably the most widely used class of models in financial applications. The GARCH model originally proposed by Bolleslev (1986) has proven to be highly successful in modelling financial data, and is arguably the most widely used class of models in financial applications. Since then, a variety of GARCH models have been proposed by various researchers, including the EGARCH model of Nelson (1991) and linear GARCH model of Taylor (1986). In the original quadratic form of the GARCH model we say that: \( u_t \) follows a GARCH\((p, q)\) process if \( u_t = \sigma_t \cdot \varepsilon_t \), where

\[
\sigma_t^2 = \beta_0 + \beta_1 \sigma_{t-1}^2 + \cdots + \beta_p \sigma_{t-p}^2 + \gamma_1 \sigma_{t-1}^2 + \cdots + \gamma_q u_{t-q}^2,
\]

and \( \varepsilon_t \) is an iid sequence of mean zero Gaussian random variables. As noted by DP, maximum likelihood estimation of this form of the GARCH model has the potential disadvantage that it is overly sensitivity to extreme returns. For example, if we consider a market crash, extreme daily absolute returns may be 10 to 20 times “normal” daily fluctuation, so the quadratic form of GARCH model yields a return effect which is 100 to 400 times the normal variance. This not only causes “overshooting” in volatility forecasting, but also carries this influence far into the future. As an alternative, Taylor suggested the following linear GARCH model: \( u_t \) follows a GARCH\((p, q)\) process if \( u_t = \sigma_t \cdot \varepsilon_t \), where

\[
\sigma_t = \beta_0 + \beta_1 \sigma_{t-1} + \cdots + \beta_p \sigma_{t-p} + \gamma_1 \sigma_{t-1} + \cdots + \gamma_q u_{t-q}.
\] (1.4)

The quadratic GARCH model seems computationally more convenient than the linear GARCH model, but linear GARCH may be more appropriate in modelling financial returns.

Another important feature in financial time series is the asymmetry in return
distributions. To take into account the asymmetric feature, asymmetric or threshold GARCH models have been proposed (see, e.g. Glosten, Jagannathan and Runkle (1993), Zakoian). Such models have been found to be adequate for capturing both the conditional heteroscedasticity and the asymmetric volatility effect in asset returns by Engle and Ng (1993) and Bekaert and Wu (2000).

To better capture both the time persistence and the asymmetric effect of the conditional volatility process, we use the linear threshold GARCH (TGARCH) model to describe firm level monthly idiosyncratic volatility. Thus, the volatility process is described by equations below,

\[ u_{i,t} = \sigma_{i,t} \varepsilon_{i,t}, \text{ where } \varepsilon_{i,t} \sim i.i.d. F_i(0,1) \]  \hspace{1cm} (1.5)

\[ \sigma_{i,t} = \beta_{i,0} + \beta_{i,1} \sigma_{i,t-1} + \gamma_{i,1} |u_{i,t-1}| + \gamma_{i,2} |u_{i,t-1}| \times I(u_{i,t-1} < 0) \]  \hspace{1cm} (1.6)

where \( \beta_{i,0}, \beta_{i,1} \) and \( \gamma_{i,1} \) are GARCH(1,1) parameters and parameter \( \gamma_{i,2} \) captures the leverage effect documented by Nelson (1991) and Engle and Ng (1993). For the \( i \)-th stock, \( F_i(\cdot) \) denotes the firm-specific cumulative density function of return innovations \( \{\varepsilon_{i,t}\}_{t=1}^{T_i} \) that can be skewed with or without excess kurtosis (fat-tails). An important feature in this model is: without making any distribution assumption on \( F_i(\cdot) \), we allow return innovations to have firm-specific underlying distributions that can vary across stocks and are better capable of capturing potential distinctive features across stocks.
1.23 A Robust Estimator of Conditional Idiosyncratic Volatility

Although the linear TGARCH structure is less sensitive to extreme returns, but it is more difficult to handle mathematically. In this paper, we propose an approach based on quantile regressions - which has some relative advantage in handling the linear structure and has the important property that it is relatively robust to distributional assumptions. See Koenker and Zhao (1996), Xiao and Koenker (2008) for studies on quantile regressions on different types of models.

Just as classical linear regression methods based on minimizing sums of squared residuals enable one to estimate models for conditional mean, quantile regression methods offer a mechanism for estimating models for the conditional quantiles. These methods exhibit robustness to extreme shocks, and facilitate distribution-free inference. Such a property is especially attractive in financial applications since many financial data such as portfolio returns or log returns are usually heavy-tailed and asymmetrically distributed.\(^3\)

Given the linear TGARCH model (1.5),

\[
\sigma_{i,t} = \beta_{i,0} + \beta_{i,1} \sigma_{i,t-1} + \gamma_{i,1} |u_{i,t-1}| + \gamma_{i,2} |u_{i,t-1}| I(u_{i,t-1} < 0) = \alpha_i' x_{i,t}
\]

where

\[
x_{i,t} = (1, \sigma_{i,t-1}, |u_{i,t-1}|, |u_{i,t-1}| 1 (u_{i,t-1} < 0))', \quad \alpha_i = (\beta_{i,0}, \beta_{i,1}, \gamma_{i,1}, \gamma_{i,2}).
\]

For standardization purpose, we assume that \( \varepsilon_i \) is iid with mean zero and unit variance. Let \( F_{t-1} \) represents information upto time \( t - 1 \), the \( \tau \)-th conditional quantile

\(^3\)Assuming quadratic specification of conditional volatility will yield similar volatility estimates.
of $y_t$ is given by

$$Q_{u_{i,t}}(\tau|\mathcal{F}_{t-1}) = H(x_{it}, \theta_i(\tau)),$$

where $\theta_i(\tau) = (\beta_i, \beta_i, \gamma_i, \gamma_i, F^{-1}(\tau))'$. We propose the following robust estimator for the conditional idiosyncratic volatility:

Step 1: We first estimate the linear TGARCH model of $u_{i,t}$ at quantile $\tau$ based on the following nonlinear quantile regression estimation:

$$\min_{\hat{\theta}_i} \sum_t \rho_\tau(u_{it} - H(x_{it}, \theta_i)).$$

we denote the estimate at the $\tau$-th quantile as $\hat{\theta}_i(\tau)$. In principle, the estimates of linear TGARCH coefficients $\beta_i$ and $\gamma_i$, and consequently the conditional volatility estimates $\sigma_{i,t}$ can be obtained from the quantile regression approach based on a particular values of $\tau$ (say, $\tau = 0.5$). To improve efficiency, the above estimation can be conducted based on a range of quantiles $\tau$, and final estimator can be obtained by appropriately aggregating information over all quantiles in Step 2. Such an aggregation is important in practice. In applications, this also helps to get rid of this numerical instability.

Step 2: Once we obtain the linear TGARCH estimates $\hat{\theta}_i(\tau)$ at different quantiles, we combine information over multiple quantiles in estimation to obtain a globally coherent estimate of the TGARCH parameters. This is accomplished most easily using minimum distance methods$^4$ as Xiao and Koenker (2008). Denote the vector of the linear TGARCH estimators $\hat{\theta}_i(\tau)$ as $\hat{\pi}_i$, i.e. $\hat{\pi}_i = [\hat{\theta}_i(\tau_1)\top, \cdots, \hat{\theta}_i(\tau_K)\top]\top$, and

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$^4$Other methods to combine information over different quantiles are possible. For example, it is possible to estimate volatility by solving a joint model across different quantiles. However, such estimation method will almost certainly lead to an insurmountable computational burden given the large number of stocks in financial applications.
let
\[ \mathbf{a}_i = [\beta_{i,0}, \beta_{i,1}, \gamma_{i,1}, \gamma_{i,2}, F_{i}^{-1}(\tau_1), \cdots, F_{i}^{-1}(\tau_K)]^\top, \]

the the relationship between \( \boldsymbol{\pi}_i \) and \( \mathbf{a}_i \) can be easily written out as a function \( \phi(\mathbf{a}_i) \), and we consider the following estimator for the vector \( \mathbf{a}_i \) that combines information over the \( K \) quantile estimates \( \hat{\theta}_i(\tau) \):
\[
\hat{\mathbf{a}}_i = \arg \min_{\mathbf{a}_i} \left( \mathbf{\hat{\pi}}_i - \phi(\mathbf{a}_i) \right)^\top A_n \left( \mathbf{\hat{\pi}}_i - \phi(\mathbf{a}_i) \right), \tag{1.7}
\]

where \( A_n \) is a positive definite weighting matrix. In this paper, we simply use the identity matrix.

Step 3. Using the TGARCH estimates from the second step, the conditional idiosyncratic volatility can then be estimated as \( \sigma_{i,t}(\hat{\beta}_i, \hat{\gamma}_i) \).

We give the statistical properties such as consistency and limiting distributions of this estimator in Appendix A. A sketch of proofs is also provide in the Appendix.

In practice, we may consider iteration of the above quantile based method. In particular, let the estimated idiosyncratic volatility from the above quantile regression be \( \sigma_{i,t}(\hat{\beta}_i, \hat{\gamma}_i) \), the estimates of return innovations \( \hat{\varepsilon}_{i,t} \) can be immediately computed as the ratio between idiosyncratic shocks \( \hat{u}_{i,t} \) and idiosyncratic volatility \( \hat{\sigma}_{i,t} \) (we denote it by \( \hat{\varepsilon}_{i,t} \)) Then, the \( \tau \)-th quantile of underlying distribution of \( \varepsilon \), i.e. \( Q_{\varepsilon_i}(\tau) \), can be updated from the empirical quantile of the ratios \( \hat{\varepsilon}_{i,t} \), this is equivalent to solving the following linear quantile regression,
\[
\min_{\xi} \sum_{t=1}^{T_i} \rho_\tau \left( \hat{\varepsilon}_{i,t} - \xi \right) \quad (1.8)
\]

With the estimated value \( Q_{\varepsilon_i}(\tau) \) of the \( \tau \)-th quantile of \( F_i(\tau) \), the GARCH coefficients are updated by solving the first non-linear quantile regression. That is, the
two quantile regressions equations (7) and (8), can be iterated to achieve finite sample improvement. In one of the appendixes, our results of Monte Carlo simulation shows that the proposed quantile regression based method is indeed capable of providing robust volatility estimates under different return distributions.

1.24 Empirical Analysis

We accede recent studies and measure idiosyncratic shocks as residuals after fitting the Fama-French (1993) three-factor model. More specifically, for the $i$-th stock, monthly excess returns are regressed on the monthly Fama-French three factor model:

$$r_{i,t} - r_{f,t} = \alpha_i + \beta_{i,MKT} MKT_t + \beta_{i,SMB} SMB_t + \beta_{i,HML} HML_t + u_{i,t}, \ t = 1, \ldots, T_t \ (1.9)$$

where $T_t$ is the number of available monthly return observations and $u_{i,t}$ is the idiosyncratic shocks estimated as residuals from fitting the market model. The linear TGARCH model is used to capture both the time persistence and the asymmetric effect of the conditional volatility process, thus, the volatility process is described by equations (1.5) and (1.6).

Given the previously described data, for the $i$-th stock, monthly idiosyncratic shocks $u_{i,t}$ are estimated as residuals after fitting the market model with all available monthly returns. The $i$-th stock conditional idiosyncratic volatilities $\sigma_{i,t}$ is then estimated using the proposed robust estimation method.

In our empirical analysis, quantile regression for individual stocks are estimated at different values of $\tau$ and the averaged conditional idiosyncratic volatility across quantiles are used to combine information over different quantiles. In particular, the linear TGARCH model is estimated at its 5%, 25%, 75% and 95% quantiles, and the
average idiosyncratic volatility across these four quantiles is used as robust estimates for the firm level conditional idiosyncratic volatility\textsuperscript{5}.

Figure 1.1: \textbf{Conditional Idiosyncratic Volatility 1964-2006}. This figure shows the value-weighted average of annualized conditional idiosyncratic volatility. The sample period is from July 1964 to December 2006.

Figure 1 plots the annualized value weighted conditional idiosyncratic volatility across stocks from July 1964 to December 2006. It can been seen that the level of idiosyncratic volatility increased to a historical high level between 1998 and 2002.

\textsuperscript{5}In this study, we estimate conditional idiosyncratic volatility using all the available return observations for individual stocks, instead of adopting a moving-window manner to estimate the conditional idiosyncratic volatility as the out-of-sample forecasts. Such practice might raise concerns about the looking-forward bias in my idiosyncratic volatility estimates. While Spiegel and Wang (2006) indeed fit EGARCH model with 60-month available data upon to month $t$, and calculate idiosyncratic volatility for month $t + 1$ as the out-of-sample forecasts; they present the same empirical findings regarding positive idiosyncratic volatility effect as in Fu (2007), where the in-sample estimates of conditional volatility is used as the measure for idiosyncratic volatility. So this looking-forward bias, if not neglectable, is not big enough to change the relation between idiosyncratic risk and return.
and returns to a rather normal level with a continuously decreasing trend starting in
2003\textsuperscript{6}.

\textsuperscript{6}A similar pattern for realized idiosyncratic volatility has been reported in Campbell et al. (2001) and Brandt et al. (2005).
1.3 Firm-Level Cross-Sectional Analysis

This section conducts a cross-sectional portfolio analysis to examine whether idiosyncratic risk has predictive power over stock returns. First, we show that the negative idiosyncratic volatility effect documented by AHXZ (2006) exists intertemporally. In other words, the portfolios containing stocks with the highest level of conditional idiosyncratic volatility earn low average returns. Second, two additional factors that previous studies (e.g. AHXZ (2006, 2007), Fu (2007), Spiegel and Wang (2006), Eiling (2006)) have failed to take into consideration may help to explain the pricing anomaly: (1) the relationship between conditional idiosyncratic volatility and returns is nonlinear; and (2) the firm level conditional skewness might also be priced. In this section, we show that many volatile stocks tend to be small and low-price stocks generally have more positively skewed returns than large high price stocks. Furthermore, "expected windfall" is proposed as an alternate measure for individual conditional skewness.

1.31 The Idiosyncratic Volatility Puzzle

To determine the contemporaneous relationship between idiosyncratic risk and returns, stocks are sorted into ten value-weighted portfolios based on the robust estimates of conditional idiosyncratic volatility. The procedure is as follows: At the end of each month, stocks are sorted based on their conditional volatility estimates for next month; we then divide the stocks evenly into 10 portfolios, such that portfolio 1 contains the 10% of stocks expected to have the lowest level of idiosyncratic volatilities over next month, while portfolio 10 contains the 10% of stocks that have the highest level of conditional idiosyncratic volatility over next month. For convenience,
we call then decile portfolios. The portfolio weights are determined to be proportional to firm market value, the product of monthly closing stock price and the number of outstanding shares. Each month, these ten decile portfolios are rebalanced. Portfolio returns as well as other useful information are then recorded.

Panel A in Table 2 presents the average conditional idiosyncratic volatility, average returns, return standard deviations, and the average market shares, average stock prices, average (log of) firm sizes, and average book-to-market ratios for ten decile portfolios over the sample period between July 1964 and December 2006. As reported in column 1 (Table 2), the average monthly conditional idiosyncratic volatilities monotonically increase from 3.9% for decile portfolio 1 to 22.41% for decile portfolio 10. Surprisingly, there are low average returns to decile portfolios containing stocks with high conditional idiosyncratic volatilities. A long-short trading strategy that longs stocks with the lowest conditional idiosyncratic volatility and shorts stocks with the highest generates an average monthly return of 1.53% from July 1964 to December 2006. This return difference is statistically significant with a t-statistic of 4.19 reported in the last row of Panel A. The cross-sectional portfolio analysis presents a strong negative intertemporal relationship between conditional idiosyncratic risk and average stock returns based on the analysis of two extreme decile portfolios. This result is exactly opposite to the positive idiosyncratic volatility effect reported by Fu (2007) and Spiegel and Wang (2006) where stocks are sorted by EGARCH estimates of conditional idiosyncratic volatility.

Moreover, Table 1 also displays distinct patterns for the average size, book-to-market ratio, and stock price of decile portfolios sorted by conditional idiosyncratic volatility. More specifically, as stocks become more volatile, the average conditional idiosyncratic volatility increases from 3.92% to 22.41%, while the average market share drops from 31.03% to less than 1%. In addition to small market capitalization,
stocks with high conditional idiosyncratic volatility tend to have low average stock prices and low book-to-market ratios. Generally, stocks with low (high) conditional idiosyncratic volatility are large (small) firms with high (low) stock prices and high (low) book-to-market ratios.

As a robustness check, the above cross-sectional portfolio analysis is repeated using only the stocks traded on the NYSE to mitigate the concern that the observed patterns are simply driven by small stocks traded on the NASDAQ and AMEX. As reported in Table 2 Panel B, the cross-section pricing anomaly remains. In particular, the return difference between decile portfolio 1 and decile portfolio 10 is $-0.83\%$ per month. This value is statistically significant with a t-statistic of $-2.83$. Therefore, excluding stocks traded on NASDAQ and AMEX changes neither the observed negative idiosyncratic volatility effect nor the fact that volatile stocks tend to have small market capitalizations, low stock prices, and low book-to-market ratios.

In summary, using the proposed robust conditional idiosyncratic volatility estimates, the cross-sectional portfolio analysis shows that the negative idiosyncratic volatility effect, documented by AHXZ (2006), exists intertemporally. That is, stocks with the highest conditional idiosyncratic volatilities yield the lowest average returns.\footnote{We also estimated the TGARCH model by the Maximum Likelihood method under the unified normality assumption on stock return innovations. The return differences between decile portfolio 10 and portfolio 1 in Table 2 turn to be positive. (The results are available upon request.) This indicates that the estimation error caused by the unrealistic assumption on individual stock return process might indeed mislead the empirical results reported by Fu (2007), Spiegel and Wang (2006).} This negative intertemporal idiosyncratic volatility effect is exactly opposite to the results of Fu (2007) and Spiegel and Wang (2006), where conditional volatility is estimated from an EGARCH model with Gaussian innovations. The idiosyncratic volatility puzzle becomes even more puzzling.
Table 1.2:  **Portfolios Sorted by Conditional Idiosyncratic Volatility**  
This table presents the raw average return (column 1) and the standard deviation (column 2) of monthly returns for the ten portfolios sorted by conditional idiosyncratic volatility. The third column shows the average market share for each portfolio. The fourth column shows the average stock price within portfolios. The sixth and seventh columns report the average log market capitalization and the book-to-market ratio of firms within portfolios. The last row in each panel, labeled '10-1', refers to the return difference between Portfolio 10 and Portfolio 1 with the t-statistic reported in square brackets. The sample period is from June 1964 through December 2006.

### Panel A: Portfolios sorted by conditional idiosyncratic volatility

<table>
<thead>
<tr>
<th>Rank</th>
<th>Average IVol(%)</th>
<th>Average Return(%)</th>
<th>Std. Dev.(%)</th>
<th>Mkt. Share</th>
<th>Stock Price</th>
<th>ln (Size)</th>
<th>B/M</th>
</tr>
</thead>
<tbody>
<tr>
<td>1 (low)</td>
<td>3.92</td>
<td>1.02</td>
<td>3.69</td>
<td>31.04</td>
<td>42.71</td>
<td>5.80</td>
<td>1.05</td>
</tr>
<tr>
<td>2</td>
<td>5.59</td>
<td>0.93</td>
<td>4.16</td>
<td>25.27</td>
<td>67.51</td>
<td>5.96</td>
<td>1.08</td>
</tr>
<tr>
<td>3</td>
<td>6.76</td>
<td>1.06</td>
<td>4.59</td>
<td>15.44</td>
<td>33.44</td>
<td>5.67</td>
<td>0.81</td>
</tr>
<tr>
<td>4</td>
<td>7.88</td>
<td>1.02</td>
<td>5.22</td>
<td>9.87</td>
<td>27.10</td>
<td>5.31</td>
<td>0.84</td>
</tr>
<tr>
<td>5</td>
<td>9.03</td>
<td>0.99</td>
<td>5.82</td>
<td>6.89</td>
<td>23.36</td>
<td>4.96</td>
<td>0.88</td>
</tr>
<tr>
<td>6</td>
<td>10.34</td>
<td>0.91</td>
<td>6.57</td>
<td>4.69</td>
<td>19.83</td>
<td>4.58</td>
<td>0.89</td>
</tr>
<tr>
<td>7</td>
<td>11.82</td>
<td>1.01</td>
<td>7.24</td>
<td>3.03</td>
<td>16.59</td>
<td>4.19</td>
<td>0.92</td>
</tr>
<tr>
<td>8</td>
<td>13.67</td>
<td>0.78</td>
<td>8.31</td>
<td>1.92</td>
<td>13.36</td>
<td>3.80</td>
<td>0.92</td>
</tr>
<tr>
<td>9</td>
<td>16.24</td>
<td>0.25</td>
<td>8.54</td>
<td>1.22</td>
<td>10.40</td>
<td>3.40</td>
<td>0.85</td>
</tr>
<tr>
<td>10 (high)</td>
<td>22.41</td>
<td>-0.51</td>
<td>9.49</td>
<td>0.63</td>
<td>6.86</td>
<td>2.83</td>
<td>0.77</td>
</tr>
<tr>
<td>10 - 1</td>
<td>-1.53</td>
<td>[-4.19]</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

### Panel B: Portfolios sorted by conditional idiosyncratic volatility (NYSE only)

<table>
<thead>
<tr>
<th>Rank</th>
<th>Average IVol(%)</th>
<th>Average Return(%)</th>
<th>Std. Dev.(%)</th>
<th>Mkt. Share(%)</th>
<th>Average Price</th>
<th>ln (Size)</th>
<th>B/M</th>
</tr>
</thead>
<tbody>
<tr>
<td>1 (low)</td>
<td>3.66</td>
<td>1.04</td>
<td>3.74</td>
<td>25.36</td>
<td>51.52</td>
<td>6.69</td>
<td>1.10</td>
</tr>
<tr>
<td>2</td>
<td>5.14</td>
<td>0.93</td>
<td>4.03</td>
<td>22.73</td>
<td>113.76</td>
<td>6.82</td>
<td>1.59</td>
</tr>
<tr>
<td>3</td>
<td>6.03</td>
<td>0.98</td>
<td>4.41</td>
<td>15.94</td>
<td>82.57</td>
<td>6.69</td>
<td>1.07</td>
</tr>
<tr>
<td>4</td>
<td>6.82</td>
<td>1.00</td>
<td>4.62</td>
<td>11.21</td>
<td>42.19</td>
<td>6.38</td>
<td>0.75</td>
</tr>
<tr>
<td>5</td>
<td>7.60</td>
<td>1.05</td>
<td>4.97</td>
<td>7.62</td>
<td>32.15</td>
<td>6.05</td>
<td>0.78</td>
</tr>
<tr>
<td>6</td>
<td>8.43</td>
<td>1.01</td>
<td>5.45</td>
<td>5.67</td>
<td>28.95</td>
<td>5.77</td>
<td>0.80</td>
</tr>
<tr>
<td>7</td>
<td>9.35</td>
<td>0.87</td>
<td>5.73</td>
<td>4.38</td>
<td>25.89</td>
<td>5.50</td>
<td>0.83</td>
</tr>
<tr>
<td>8</td>
<td>10.49</td>
<td>0.90</td>
<td>6.41</td>
<td>3.44</td>
<td>22.38</td>
<td>5.24</td>
<td>0.83</td>
</tr>
<tr>
<td>9</td>
<td>12.21</td>
<td>0.99</td>
<td>7.32</td>
<td>2.29</td>
<td>18.83</td>
<td>4.88</td>
<td>0.83</td>
</tr>
<tr>
<td>10(high)</td>
<td>16.90</td>
<td>0.21</td>
<td>8.10</td>
<td>1.36</td>
<td>13.91</td>
<td>4.33</td>
<td>0.78</td>
</tr>
<tr>
<td>10 - 1</td>
<td>-0.83</td>
<td>[-2.83]</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
1.32 Idiosyncratic Volatility Effect: More than a Linear Effect

One of the observations in this study is that the effect of idiosyncratic volatility on stock returns is not linear. As shown in Table 2 Panel A, the average returns for decile portfolios sorted on conditional idiosyncratic volatility are steady and even slightly increasing within the first 7 decile portfolios of stocks with a relatively lower level of idiosyncratic risk. The average portfolio returns drop precipitously as the level of idiosyncratic risk increases even higher among decile portfolios 8, 9 and 10, which merely comprise 3.77\% of total market capitalization of stocks traded on major markets. And the drops in average returns among the last three decile portfolios eventually lead to a large negative return difference of $-1.53\%$ per month between the two extreme decile portfolios.\textsuperscript{8} In addition, this non-monotonicity of idiosyncratic volatility becomes even clearer after excluding the relatively small stocks traded on the NASDAQ and AMEX. As shown in Table 1 Panel B, the observed large negative return difference is simply driven by the drop in the last decile portfolio that comprises less than 1.5\% of total market capitalization.

Figure 2 plots average returns versus average idiosyncratic volatility for decile portfolios sorted on conditional idiosyncratic volatility. The black line plots the relationship for portfolios formed with stocks traded on NYSE, NASDAQ and AMEX, while the red line plots the relationship for stocks traded only on the NYSE. It presents a nonlinear effect of idiosyncratic volatility on stock returns, and the nonlinearity is robust after excluding stocks traded on AMEX and NASDAQ. Given the fact that the portfolios with an excessively high level of conditional idiosyncratic volatility only

\textsuperscript{8}The same pattern also exists in AHXZ (2006). In AHXZ (2006, Table VI B), the average return increases from 1.04\% for quintile 1 to .2\% for quintile 3 and plunges to $-0.02\%$ for quintile 5. This leads to a large negative difference in average returns between quintile portfolios 5 and 1.
Figure 1.2: **Portfolio Return vs. Idiosyncratic Volatility.** The figure shows the relationship between average returns and average conditional idiosyncratic volatilities for the decile portfolios sorted by conditional idiosyncratic volatility. The solid line represents the portfolio analysis using all stocks traded on NYSE, NASDAQ and AMEX; while the dashed line shows the relationship using stocks traded on NYSE only.
contribute to a small proportion of the total market share, the cross-sectional portfolio analysis focusing exclusively on the two extreme portfolios may not be able to fully reveal the true effect of idiosyncratic volatility. In fact, as reported in Panel A and B of Table 2, the idiosyncratic volatility effect is ambiguous within the first seven decile portfolios, which comprise over 95% of the total market share of stocks traded on the three major markets and on NYSE only. However, previous studies have not considered this nonlinear effect of idiosyncratic volatility on stock returns.

1.33 Firm-Level Conditional Skewness — Expected Windfall

Table 2 also presents another intriguing result. That is, stocks with high idiosyncratic risk are relatively small stocks with low stock prices. As shown in Panel A of Table 2, stocks in decile portfolio 10 have an average conditional idiosyncratic volatility of 22.41% per month and an average stock price of $6.86 per share. In addition, the 10% riskiest stocks comprise only 0.63% of total market capitalization on average.

Given the fact that share holders have limited obligation, stock return distribution by nature is positively skewed. Moreover, Chen, Hong and Stein (2001) and Duffee (2002a, b) show that small stocks have more positively skewed returns than large stocks. Low price stock return distributions should also skew to the right, given the already low stock prices. Intuitively, given the very limited downside risk of a stock price that is already low, investors may decide to hold low price stocks as out-of-the-money options or simply as lotteries, if they are expected to realize extraordinary returns. As a result, individual skewness especially for low price small stocks

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The similar fact between stock price and realized idiosyncratic volatility is also reported by Brandt et al. (2005). They show that high realized idiosyncratic volatility is concentrated in low-price stocks with a stock price less than $10. They also confirm that such a distinctive feature of idiosyncratic volatility for high- and low-price stocks is robust after controlling for other explanatory factors, including firm’ size er al. However, the study of Brandt et al. (2005) aims mainly to explain the historical time variation in aggregated realized idiosyncratic volatility over 1998 to 2002.
could play a very important role in determining future returns. Barberis and Huang (2007) study the implications of the cumulative prospect theory (CPT) of Tversky and Kahneman (1992) on security prices. They present a theoretical model where investors overweight the tail probabilities and take a substantial undiversified position in securities that exhibit lottery-like characteristics and are expected to generate windfall gains. They show that such lottery-like securities may be over-priced and subsequently yield low returns. Therefore, low returns for volatile stocks might be partially explained by large positive conditional skewness.

Empirically, it is difficult to construct an accurate measurement for the firm level conditional skewness that can be used to identify stocks having a small chance of "making it big." Zhang (2006) argues that past skewness is a poor candidate for such a measure due to data constraints and the time-varying property of individual skewness.\(^{10}\) He then employs the realized intra-industry skewness as an alternative measure for firm level conditional skewness for all of the stocks within the same industry. Zhang (2006) also provides some evidence that stocks with high realized intra-industry skewness might be overpriced and expect low returns. However, by assuming companies within the same industry are equally likely to become successful, intra-group skewness has its limitations. The intra-group measure is particularly problematic since it assumes all firms within the same industry, regardless of size, have the same return skewness. On the other hand, Chen, Hong and Stein (2001) and Duffee (2002a, b) show that the firm size is the most important determinant for individual skewness. In addition, the intra-group skewness might not be appropriate for stocks in industries of considerable competition where one company’s breakthrough might be another’s downfall. Moreover, since the intra-group measurement is calculated

\(^{10}\)Some conditional skewness models, e.g. Harvey and Siddique (1999), might not be a good alternative for posing strong distribution assumption on stock return process.
using raw returns, it might be contaminated by market or industry factors. For these reasons, a better measure for firm level conditional skewness is needed.

We propose "expected windfall" as an alternative measure for conditional skewness to quantify the upside potential of stock return distributions. The expected windfall measurement is an analogue to the expected shortfall (Artzner et al. 1997, 1999). It focuses exclusively on the upper tail of stock return distribution, where the most profitable and most unlikely returns occur. For a random variable $x$ with a distribution function $F_x(\cdot)$, the expected windfall, denoted as $EW_x(\tau)$ is defined by equation (9) as the conditional expectation of $x$, given that the realized outcome is among the best $100(1 - \tau)%$ of all possible outcomes for any $\tau$ in the interval $(0, 1)$,

$$EW_x(\tau) \equiv \frac{1}{1 - \tau} E[x|x \geq F_x^{-1}(\tau)]$$ (1.10)

where $F_x^{-1}(\tau)$ is the $\tau - th$ quantile of the distribution of $x$. For a high value of $\tau$ of 0.95, $EW(0.95)$ directly measures the expected value of $x$ given that the outcome is among the best 5% of all possible outcomes. As a direct application of Theorem 2 in Bassett, Koenker and Kordas (2006), equation (9) is equivalent to the following optimization problem that leads immediately to a linear quantile regression:

$$EW_x(\tau) \equiv \frac{1}{1 - \tau} \min_\xi E\rho_\tau(x - \xi) - \frac{\tau}{1 - \tau} Ex,$$ (1.11)

where $\rho_\tau(\cdot)$ is the check function such that $\rho_\tau(x) = x(\tau - I(x < 0))$ for $0 < \tau < 1$. Empirically, the value of the expected windfall for the $i$-th stock in month $(t + 1)$ can be estimated as follows

$$\hat{EW}_{u_i,t}(\tau) \equiv \frac{1}{1 - \tau} \min_\xi \sum_{s=1}^t \rho_\tau(u_{i,s} - \xi) - \frac{\tau}{1 - \tau} \sum_{s=1}^t u_{i,s},$$ (1.12)
where the time series of \( \{ \tilde{u}_{i,s} \}_s^{t} \) contains all available idiosyncratic shocks for the i-th stock at the end of month \( t \).

Expected windfall directly measures firm level expected returns when "winning the lottery" and as such, the realization of the most profitable but uncommon idiosyncratic shocks. The expected windfall has the computational advantages as the value of EW can be estimated by solving a quantile regression problem. Therefore, the expected windfall might serve as a more accurate alternative measurement for conditional skewness of individual stock returns.

To summarize, this section reveals that the negative idiosyncratic volatility effect documented by AHXZ (2006) exists intertemporally. In other words, the portfolios containing stocks with the highest level of conditional idiosyncratic volatility earn low average returns. Furthermore, two factors that previous studies have failed to take into consideration may help to explain the pricing anomaly: (1) the relationship between conditional idiosyncratic volatility and returns is nonlinear; and (2) the firm-level conditional skewness might be priced. The "expected windfall" is proposed as an alternate measure for individual conditional skewness. We further investigate the nonlinear effect and the impact of expected windfall in the next section.
1.4 Firm-Level Cross-sectional Regression Tests

The portfolio analysis described in section 3 is a straightforward method to study the relationship between idiosyncratic risk and returns; however, one cannot effectively control for other risk factors. Also the observed negative contemporaneous idiosyncratic volatility effect might be driven by the two extreme portfolios with information in other portfolios being completely discarded. Using Fama-MacBeth (1973) cross-sectional regressions, this section investigates the relationship between conditional idiosyncratic volatility and expected returns with a wide range of control variables and identifies a positive idiosyncratic volatility effect for stocks comprising over 99% of the total market capitalization.

The sample runs from July 1964 to December 2006, encompassing 510 months, and includes stocks traded on the NYSE, AMEX and NASDAQ. The individual stock returns, outstanding share numbers, and average monthly trading volumes are from the CRSP, and the annual book equity for individual firms are from the COMPUSTAT database. The systematic risk measures, including BETA, the market value of equity (size), and the book-to-market ratio (BE/ME), are constructed as described by Fama and French (1993), and later by Fu (2007). The momentum effect (Jegadeesh and Titman (1993)) in month t is captured by the geometric average return from month (t – 7) to month (t – 2).

The Fama-MacBeth regression test is conducted in two stages. In the first stage, for each month from July 1964 to December 2006, the cross-sectional stock returns are regressed on potential determinants including conditional idiosyncratic volatility, the market beta, the log of firm size, the book-to-market ratio, lagged returns. The returns and explanatory variables are weighted by the square root of the firm’s market capitalization at the end of the previous month. By forcing the
model to better fit observations of large stocks, the idiosyncratic volatility effect examined here is less likely to be driven by small stocks. For each month $t$, where $t = 1, \ldots, 510$, the following least square regression uses the square root of the firms’ market capitalization in month as weights,

$$r_{i,t} = \gamma_{0,t} + \gamma_{1,t}\sigma_{i,t} + \sum_{k=2}^{K} \gamma_{k,t}X_{i,k,t} + \nu_{i,t}, \ i = 1, \ldots, N_t \quad (1.13)$$

Where $r_{i,t}$ is the return on the $i$–th stock in month $t$, $\sigma_{i,t}$ is the robust estimates of conditional idiosyncratic volatility, and $X_{i,k,t}$ are other explanatory variables including BETA, firm size, book-to-market ratio, etc. In the second stage, with the time-series of regression coefficients obtained from the first stage, I calculate the sample mean and the Newey-West (1989) $t$-statistic for each coefficient. The robust t-statistic is used to test whether the corresponding coefficient is statistically different from zero, and to determine whether the intended variable has significant explanatory power on cross-sectional expected returns holding other variables constant.

Table 3 reports the results of the Fama-MacBeth regressions including regression coefficients and robust t-statistics in brackets. Each row of the table presents models with different sets of potential explanatory variables for cross-sectional expected returns.

1.41 Nonlinearity in Idiosyncratic Volatility Effect

Including only conditional idiosyncratic volatility, market beta, firm size, book-to-market ratio, and lagged return, Model (1) suggests an ambiguous idiosyncratic volatility effect, since the mean of the coefficient in front of conditional idiosyncratic volatility estimates is 0.02 and statistically insignificant with a t-statistic of 0.085.
While Fu (2007) reports a significant positive coefficient in front of the EGARCH estimates conditional idiosyncratic volatility, the result further suggests that the positive idiosyncratic effect of Fu (2007) might be driven by the estimation error in the Gaussian MLE.

Motivated by the nonlinearity of idiosyncratic volatility effect demonstrated in Table 2 and Figure 2, the existence of the nonlinear effect is tested by including a high-order polynomial of conditional idiosyncratic volatility into the cross-sectional regressions:

\[ r_{i,t} = \gamma_{0,t} + \sum_{l=1}^{L} \gamma_{l,t} \sigma_{i,t}^l + \sum_{k=L+1}^{K} \gamma_{k,t} X_{i,k,t} + \nu_{i,t}, \quad i = 1, \ldots, N_t \]  

(1.14)

Results of Model (2) confirm the existence of the nonlinear idiosyncratic volatility effect. The first two moments, idiosyncratic volatility and idiosyncratic variance, both have statistically significant predictive power on the cross-sectional expected returns and are jointly significant with an F-statistic of 50.3.\(^{11}\)

The average effect of conditional idiosyncratic volatility on expected return is equal to

\[ \frac{\partial \hat{\gamma}_i}{\partial \sigma_i} = \hat{\gamma}_1 - 2\hat{\gamma}_2 \sigma_i, \]  

(1.15)

where \( \hat{\gamma}_1 \) and \( \hat{\gamma}_2 \) are the average regression coefficients in front of conditional idiosyncratic volatility and variance estimates. For any stock \( i \), as long as:

\[ \sigma_i < \frac{\hat{\gamma}_1}{2\hat{\gamma}_2}, \]  

(1.16)

then the value of the partial derivative of expected return with respect to conditional idiosyncratic volatility \( \partial \hat{\gamma}_i / \partial \sigma_i \) would be positive, indicating a positive idiosyncratic idiosyncratic variance.

\(^{11}\)The coefficients of third and higher-order terms are statistically insignificant.
Table 1.3: **Value-Weighted Fama-MacBeth Regression Results**

For each month in the sample period, the raw returns are regressed on intended explanatory variables. The column labeled $F$ reports the average F-statistic used to conduct the joint test on the coefficients before the first and second order of conditional idiosyncratic volatility. Robust Newey-West (1987) t-statistics are in square brackets under the coefficients.

<table>
<thead>
<tr>
<th></th>
<th>$\text{const}$</th>
<th>$\sigma$</th>
<th>$\sigma^2$</th>
<th>$EW_{95%}$</th>
<th>$BET\alpha$</th>
<th>ln($size$)</th>
<th>$B/M$</th>
<th>$ret_{-2:-7}$</th>
<th>$RVol$</th>
<th>$ret_{-1}$</th>
<th>$F$</th>
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<td>0.172</td>
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<td>0.73</td>
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<td></td>
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<td>[43.9]</td>
<td></td>
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<tr>
<td>(2)</td>
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<td>-1.798</td>
<td>0.168</td>
<td>0.013</td>
<td>0.002</td>
<td>0.712</td>
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<td></td>
<td></td>
<td>50.33</td>
</tr>
<tr>
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<td>[2.33]</td>
<td>[7.63]</td>
<td>[-10.0]</td>
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<td>[19.1]</td>
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<td>[44.0]</td>
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<tr>
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<td>-0.123</td>
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<td>[2.0]</td>
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<td></td>
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<tr>
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<td>-0.126</td>
<td>0.164</td>
<td>0.002</td>
<td>0.712</td>
<td></td>
<td>-0.05</td>
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<td>[2.34]</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>(5)</td>
<td>0.081</td>
<td>0.684</td>
<td>-1.57</td>
<td>-0.124</td>
<td>0.140</td>
<td>0.002</td>
<td>0.679</td>
<td></td>
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<td>[2.12]</td>
<td></td>
<td>[45.3]</td>
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</table>


volatility effect. Substitute the regression coefficients and the effect of conditional idiosyncratic volatility on the expected return will be positive so long as conditional idiosyncratic volatility is less than 9.28% (annualized 32.15%). As stocks become more volatile with a level of monthly conditional idiosyncratic volatility higher than 9.28%, the idiosyncratic volatility effect is reversed as the value of the partial derivative \( \partial \hat{r}_i / \partial \sigma_i \) becomes negative. Given the existence of the second-order term of idiosyncratic volatility, the cross-sectional expected return would drop precipitously as the level of volatility goes higher. This effect might help explain why there is a sharp drop in returns among the most volatile stocks, as shown in Figure 2.

Only a small proportion of stocks (less than 15% of total market capitalization) have a level of conditional idiosyncratic volatility higher than 9.28%. Results in Table 3 Model (2) reveal a positive idiosyncratic volatility effect for stocks comprising over 85% of total market capitalization of stocks traded in the three major markers. This positive effect reverses among volatile stocks that have an average level of idiosyncratic volatility higher than 9.28%.

All other regression coefficients have meaningful magnitude and signs in the above regression test: the coefficient of BETA is positive and statistically significant. The coefficient of the firm (log) market capitalization is positive and significant, consistent with Merton’s (1973) theoretical prediction that larger firms earn higher returns. In addition, the book-to-market ratio is positively related to the average cross-sectional returns, implying that value firms tend to have higher returns than growth firms. Moreover, the results confirm a positive momentum effect.
1.42 Individual Skewness—Expected Windfall

Our analysis in the previous section indicates that investors should expect a high return for bearing idiosyncratic risk, as long as the level of idiosyncratic volatility is less than 9.28% per monthly (annualized 32.15%). As shown in Table 2 this positive idiosyncratic volatility effect exists for stocks comprising over 85% of total market capitalization. The negative idiosyncratic volatility effect only concentrates among the most volatile, small, low price stocks. These stocks comprise less than 15% of total market capitalization.

As argued in Section 3C, small low price stocks tend to have higher positive skewness than large high price stocks, and they are more likely to generate windfall profits. As predicted by the theoretical model developed by Barberis and Huang (2007), lottery-like securities may be over-priced and are expected to yield low returns. Hence, if individual conditional skewness is also priced, it might help to explain why stocks with high idiosyncratic volatility have low returns. "Expected windfall" was introduced as an alternative measure for individual conditional skewness.

We next look at the influence of individual skewness by including the "expected windfall" in the cross-sectional regression to control for individual stock conditional skewness. As shown in Table 3 Model (3), expected windfall has a significant predictive power on expected returns. The average regression coefficient in front of EW is $-0.12$ with a robust $t$-statistic of $-9.58$. This negative coefficient is consistent with the prediction of Barberis and Huang (2007), since a large value of EW would indicate a similarity of stock return distribution to that of a lottery. The stock may be over priced and consequently earn low expected returns.

More importantly, after controlling for EW the average value of the regression coefficient for conditional idiosyncratic volatility is almost doubled while that of con-
ditional idiosyncratic variance stays roughly the same. As a result, there would be a positive idiosyncratic volatility effect \((\partial r_i / \partial \sigma_i > 0)\) for any value of idiosyncratic volatility smaller than 18.77% (annualized 65%) while holding expected-windfall and other variables constant. Only a very small proportion of stocks (comprising less than 1% of the total market capitalization) has an average level of conditional idiosyncratic volatility higher than 18.77%. The results presented in Table 3 Model (3) reveal a positive relationship between conditional idiosyncratic volatility and expected returns for stocks comprising over 99% of total market capitalization of major markets.

The rest of stocks are "penny-like" stocks, which are the most volatile, low-price, small stocks with an average monthly idiosyncratic volatility higher than 18.77%, an average price of $7.53 per share, and an average market capitalization of $93 million. Since these "penny-like" stocks are generally not in the main scope of financial studies, we conclude that there is a positive relationship between conditional idiosyncratic volatility and expected stock returns, which is consistent with the predictions of most finance theories.

As shown in Table 3 from Models (4) and (5), the above results are robust after controlling for other known risk factors for cross-sectional stock returns including distress risk measurement (Altman’s (1968) Z-score and Ohlson’s (1980) O-score), trading volume (Gervais et al., 2001), leverage effect (measured as the ratio of total book value of assets to book value of equity, AHXZ(2006)) and liquidity (measured as the relative volume defined as the ratio between share volume and share outstanding as in Burton and Xu (2006)).

In summary, with the present robust estimates of conditional idiosyncratic volatility, controlling for the nonlinear idiosyncratic volatility effect and firm level conditional skewness, we find a positive relationship between conditional idiosyncratic volatility and cross-sectional expected returns for stocks comprising over 99% of total
market capitalization. This positive idiosyncratic volatility effect is consistent with predictions of financial theories.
1.5 Firm-Level Cross-Sectional Quantile-Regression Test

While the existing literature focuses on the central tendency of the idiosyncratic volatility effect, this section goes one step further to investigate the predictive power of idiosyncratic volatility and other risk factors on expected returns across the conditional distribution of cross-sectional returns. Results from the Fama-MacBeth tests in Section 4 have identified a positive idiosyncratic volatility effect on cross-sectional expected returns at the average level. Such a positive relationship reverses for only a small portion of "penny-like" stocks that are extremely volatile with an average level of monthly idiosyncratic volatility higher than 18.77% and comprising less than 1% of the total market share.

Using the conventional Fama-MacBeth (1973) test, the mean-level idiosyncratic effect reflects the relationship between idiosyncratic risk and return for firms that expect to earn the mean of cross-sectional stock returns. This approach implicitly assumes that possible differences in terms of the impact of conditional idiosyncratic volatility along the conditional distribution of returns are unimportant. This mean effect may not, however, imply that conditional idiosyncratic volatility has the same predictive power on expected returns for both the under-performing and over-performing stocks. For example, one would reasonably expect that bearing idiosyncratic risk for underperforming stocks should not be very rewarding since high idiosyncratic risk may just reflect some intrinsic drawbacks. The quantile regression is a powerful tool that characterizes the entire distribution of the dependent variable given a set of regressors. Instead of only describing the mean tendency of idiosyncratic risk effect as with least squares regression, a cross-sectional quantile-regression
test that is able to reveal the idiosyncratic volatility effect at each possible value of cross-sectional stock returns is proposed. This cross-sectional quantile-regression test might also be viewed as a more sophisticated cross-sectional portfolio analysis with stocks sorted into hundreds of portfolios based on their returns. The idiosyncratic volatility effect within each portfolio is pinpointed while controlling for other risk factors.

Furthermore, the conventional Fama-MacBeth regression test, though a standard methodology in the financial literature, has long been criticized for having low power and suffering from estimation errors, lack of independence, and homoscedasticity among cross-sectional returns. Compared with the method of least squares, quantile regression has been proven a more robust method in the presence of heteroscedasticity and error-in variables. As demonstrated by Barnes and Hughes (2002) quantile regression analysis of cross-sectional stock returns alleviates some statistical problems that might plague the conventional Fama-MacBeth regression test.

Similar to the Fama-MacBeth regression test from Section 4, the cross-sectional quantile-regression test is conducted in the two stages. In the first stage, for each month from July 1964 to December 2006, a weighted cross-sectional quantile-regression is conducted at different $\tau$, the quantile of the conditional distribution of cross section stock returns for $\tau \in (0, 1)$.

$$r_{i,t}(\tau) = \gamma_{0,t}(\tau) + \gamma_{1,t}(\tau) \sigma_{i,t} + \sum_{k=2}^{K} \gamma_{k,t}(\tau) X_{i,k,t} + \nu_{i,t}(\tau), \ i = 1, ..., N_t. \quad (1.17)$$

where $r_{i,t}$ is the return on the $i-th$ stock in month $t$, $\sigma_{i,t}$ is the estimate of conditional idiosyncratic volatility in the same month, and $X_{i,k,t}$ are other explanatory variables including expected windfall, BETA, firm size, book-to-market ratio, etc. The returns
and explanatory variables are weighted by the square root of the firm market capitalization at the end of previous month. In the second stage, with the time-series of regression coefficients from the first stage, the sample mean and the Newey-West (1989) t-statistic for each coefficient are calculated at different quantiles. The robust t-statistic then tests whether the variable has significant predictive power on expected returns at certain quantiles of the conditional distribution of cross-sectional returns.

Table 4 reports the average coefficients and the corresponding robust t-statistics of the cross-sectional quantile regression conducted at the 1, 10, ..., 90, 99 percentile of the conditional distribution of cross-sectional stock returns. Figure 3 plots the average coefficients of conditional idiosyncratic volatility at different quantiles of the cross-sectional return distribution.

![Graph](image)

Figure 1.3: **Quantile-Dependent Idiosyncratic Volatility Effect.** This figure reveals a positive relationship between conditional idiosyncratic volatility and expected return for 60% of stocks, which comprise 98.8% of the total market capitalization of NYSE, NASDAQ and AMEX.

An intriguing result uncovered by this cross-sectional quantile regression test
Table 1.4: Firm-Level Cross-Sectional Quantile Regression Test
For each month in the sample period, the raw returns are regressed on the intended explanatory variables at different percentiles of the cross-sectional stock return distribution. Robust Newey-West (1987) t-statistics are in square brackets under the coefficients.

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<th>τ</th>
<th>const</th>
<th>σ</th>
<th>BETA</th>
<th>ln(size)</th>
<th>B/M</th>
<th>ret_{-2,-7}</th>
<th>EW_{95%}</th>
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<td>1%</td>
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<td>0.024</td>
<td>0.003</td>
<td>0.615</td>
<td>0.548</td>
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<td>[29.4]</td>
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</tr>
<tr>
<td>10%</td>
<td>0.073</td>
<td>-1.31</td>
<td>0.159</td>
<td>0.019</td>
<td>0.002</td>
<td>0.658</td>
<td>0.206</td>
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<td>[2.34]</td>
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<tr>
<td>20%</td>
<td>0.083</td>
<td>-0.682</td>
<td>0.159</td>
<td>0.018</td>
<td>0.002</td>
<td>0.672</td>
<td>0.07</td>
</tr>
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<td>[1.91]</td>
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<tr>
<td>30%</td>
<td>0.09</td>
<td>-0.269</td>
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<td>0.002</td>
<td>0.681</td>
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<td>40%</td>
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<td>0.094</td>
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<td>[1.73]</td>
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<td>50%</td>
<td>0.104</td>
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<td>60%</td>
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<td>70%</td>
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<td>80%</td>
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<td>0.016</td>
<td>0.002</td>
<td>0.718</td>
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<td>[19.23]</td>
<td>[1.97]</td>
<td>[46.32]</td>
<td>[-23.8]</td>
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<td>90%</td>
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<td>0.015</td>
<td>0.001</td>
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<td>99%</td>
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is the opposite idiosyncratic volatility effect for underperforming and outperforming stocks. For all \( \tau \) less than 33\%, the average coefficient of idiosyncratic volatility is negative and significant, indicating a negative relationship between conditional idiosyncratic volatility and expected returns among underperforming stocks. These 33\% of stocks comprise only 0.8\% of the total market capitalization of NYSE, NASDAQ, and AMEX. While for the upper-tail 60\% of stocks, the average coefficient of idiosyncratic volatility is positive, indicating a positive idiosyncratic volatility effect for stocks comprising 98.75\% of total market share of three major markets. The rest stocks that comprise 1.05\% of market share have an ambiguous idiosyncratic volatility effect, as the average value of \( \gamma_1(\tau) \) is insignificant for any value of \( \tau \) larger than 33\% and smaller than 40\%.

Hence, the firm-level cross-sectional quantile regression test confirms the existence of a positive idiosyncratic volatility effect for the majority of stocks comprising about 99\% of the total market share of NYSE, NASDAQ and AMEX. Furthermore, the cross-sectional quantile-regression test shows that this positive idiosyncratic effect is also significant. Once again, the negative idiosyncratic volatility effect exists only among small-cap underperforming stocks, which account for 33\% of the stocks and comprise less than 1\% of the total market capitalization.

The firm-level cross-sectional quantile regression reveals that the average coefficient of the conditional idiosyncratic volatility is indeed a function of \( \tau \), the quantile of cross-sectional stock return distribution. Such a location shift in the regression coefficient of conditional idiosyncratic volatility could be a result of heteroscedasticity in the cross-sectional quantile regression. A robustness check in the Appendix B, however, rules out the potential heteroscedasticity as an explanation for the observed quantile-dependent idiosyncratic volatility effect.
1.6 Concluding Remarks

This paper analyzes the roles of idiosyncratic risk and individual stock conditional skewness in determining cross-sectional stock returns and makes two main contributions to the literature on idiosyncratic risk. For the first contribution, I show that the assumption of Gaussian innovations in EGARCH specification is firmly rejected for over 90% of stocks traded on major markets. Consequently, the EGARCH estimates of conditional idiosyncratic volatility may involve significant estimation errors that might mislead the empirical results of Fu (2007) and Spiegel and Wang (2005). I therefore develop a novel method to robustly estimate conditional idiosyncratic volatility in a quantile regression framework. Most importantly, in my setting, individual stocks are allowed to have firm-specific return distributions that can vary from stock to stock. Hence, the present model would prove more capable of better capturing distinctive features among different stock return processes. While the estimates of conditional idiosyncratic volatility would be immune from estimation errors caused by unmodeled skewness and kurtosis, they are also robust to a potentially different return process across stocks.

In my second contribution, I show that both idiosyncratic variance (the nonlinear effect) and individual conditional skewness have significant predictive power on cross-sectional expected returns and help to explain the idiosyncratic volatility puzzle. In addition, "expected windfall" is introduced as an alternative measure for the conditional skewness of individual stock returns. After controlling for the nonlinear effect and individual conditional skewness, a major piece of the idiosyncratic volatility puzzle is solved. Specifically, the Fama-MacBeth regression tests identify a positive intertemporal idiosyncratic volatility effect on cross-sectional expected returns among stocks comprising about 99% of 99% of the total market capitalization of the NYSE,
NASDAQ and AMEX. Furthermore, my cross-sectional quantile regression test confirms that this positive relationship between conditional idiosyncratic volatility and expected returns is also statistically significant.

The puzzle of a negative intertemporal idiosyncratic volatility effect only remains among "penny-like" stocks that are the most volatile, micro-cap, low price, underperforming stocks, and comprise only 1% of total market capitalization. Since these "penny-like" stocks may very likely be on the edge of bankruptcy and failure\textsuperscript{12}, a promising topic for future research would be to control for distress risk while examining the negative intertemporal idiosyncratic volatility effect among these "penny-like" stocks.

\textsuperscript{12}Campbell, Hilscher and Szilagyi (2008) show that firms with lower profitability, lower market capitalization, lower past stock returns, more volatile past stock returns, and lower prices per share are more likely to file for bankruptcy, be delisted, or receive a D rating.
1.7 References


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1.8 Appendix A: Properties of The Estimator

Notice that the linear TGARCH model has the following variance structure:

\[ u_{i,t} = \sigma_{i,t} \varepsilon_{i,t}, \text{ where } \varepsilon_{i,t} \overset{iid}{\sim} F_i(0, 1) \]

where

\[ \sigma_{i,t} = \sigma_{i,t}(\alpha_i) = \beta_{i,0} + \beta_{i,1} \sigma_{i,t-1} + \gamma_{i,1} |u_{i,t-1}| + \gamma_{i,2} |u_{i,t-1}| I(u_{i,t-1} < 0) = \alpha_i' x_{i,t} \]

and the \( \tau \)-th conditional quantile function of \( u_{i,t} \) is given by

\[ Q_{u_{i,t}}(\tau | F_{t-1}) = H(x_{i,t}, \theta_i(\tau)). \]

For convenience of asymptotic analysis, we introduce the following regularity assumptions.

**ASSUMPTIONS**

A1. \( H(x, \theta) \) is a measurable function in \( x \) for each \( \theta \in \Theta \), where \( \Theta \) is a compact subset of the Euclidean space.

A2. Let \( \dot{H}_\theta(x_{i,t}, \theta) = \partial H(x_{i,t}; \theta) / \partial \theta \), \( \Omega_i(\tau) \) are non-singular for any \( i \), where

\[ \Omega_i(\tau) = \mathbb{E} \left[ \dot{H}_\theta(x_{i,t}, \theta_i(\tau)) \dot{H}_\theta(x_{i,t}, \theta_i(\tau))' \right] \]

A3. For each \( i, \varepsilon_{i,t} \) are iid with mean zero and unit variance. The density of \( \varepsilon_{i,t}, f_i(\varepsilon) \) is continuous and bounded from above. There exist \( \epsilon_1 > 0 \) and \( p > 0 \) such that

\[ P \left( \frac{1}{\sigma_t} f_i \left( F_i^{-1}(\tau) \right) \geq \epsilon_1 \right) \geq p, \]

A4. \( u_{i,t}(\theta) = \sigma_{i,t} \varepsilon_{i,t} \) are stationary \( \beta \)-mixing with mixing decay rate \( \beta_t = O(b^{-t}) \) for some \( b > 1 \).
A5. For each $i$, there exist a measurable function $a_{it}$ such that for all $\theta$ and $t, n, |H(x_{it}, \theta)| \leq a_{it}$, where $E|a_{it}|^{r_0} \leq \Delta < \infty$ for some $r_0 > 1$.

We give the statistical property of the estimators in each step in this Appendix. A sketch of proof is given in Appendix B.

Step 1 estimates the linear TGARCH model of $u_{i,t}$ at quantile $\tau$ based on nonlinear quantile regression estimation. Theorem 1 below gives the asymptotic property of the quantile regression TGARCH estimator.

**Theorem 1:** Under Assumptions A1 - A5, for any $\tau \in (0, 1)$, (1) $\hat{\theta}_i(\tau) = \theta_i(\tau) + o_p(1)$; (2)

$$\sqrt{n} \left( \hat{\theta}_i(\tau) - \theta_i(\tau) \right) \Rightarrow N(0, \Sigma_i(\tau)),$$

where

$$\Sigma_i(\tau) = \tau(1-\tau)V_i(\tau)^{-1}\Omega_i(\tau)V_i(\tau)^{-1}$$

$$V_i(\tau) = E \left[ f_{u|x_i}(Q_{u_{i,t}}(\tau|x_{i,t})) \hat{H}_\theta(x_{i,t}, \theta_i(\tau)) \hat{H}_\theta(x_{i,t}, \theta_i(\tau))^\top \right]$$

$$\Omega_i(\tau) = E \left[ \hat{H}_\theta(x_{i,t}, \theta_i(\tau)) \hat{H}_\theta(x_{i,t}, \theta_i(\tau))^\top \right].$$

Using the result of Theorem 1, we can obtain the limiting distribution for the vector $\widehat{\pi}_i = \left[ \hat{\theta}_i(\tau_1)^\top, \cdots, \hat{\theta}_i(\tau_K)^\top \right]^\top$.

Step 2 combines information over multiple quantiles to obtain a globally coherent estimate of the TGARCH parameters. Theorem 2 gives the asymptotic distribution of the minimum distance estimator.

**Theorem 2:** Under Assumptions A1 - A5, let $A = \lim_{n \to \infty} A_n$, and $G_i = \partial \phi(a_i)/\partial a_i^\top$, then

$$\sqrt{n}(\hat{a}_i - a_i) \Rightarrow N(0, \Pi_i)$$

where $\Pi_i = \left[ G_i^\top A G_i \right]^{-1} G_i^\top A \Sigma_i G_i \left[ G_i^\top A_n G_i \right]^{-1}$, and $\Sigma_i = \text{diag} \left[ \Sigma_i(\tau_1), \cdots, \Sigma_i(\tau_K) \right]$.

The conditional idiosyncratic volatility is estimated in Step 3 using the esti-
mates from the second step. The following Theorem gives its limiting distribution.

**THEOREM 3:** Under Assumptions A1 - A5, let \( \tilde{\sigma}_{i,t} = \sigma_{i,t}(\tilde{\beta}_i, \tilde{\gamma}_i) \), \( \underline{x}_{i,t} = A_i(L)x_{i,t} \), where \( A_i(L) = 1 - \beta_{i,1}L \), denote the sub-matrix of \( \Pi_i \) corresponding to \((\tilde{\beta}_i, \tilde{\gamma}_i)\) as \( \Pi_{i\alpha} \), and define \( \omega_{i,t}^2 = \underline{x}_t' \Sigma_{i\alpha}(\tau) \underline{x}_t \), then

\[
\sqrt{n}(\tilde{\sigma}_t - \sigma_t) \Rightarrow N(0, \omega_{i,t}^2).
\]
1.9 Appendix B: A Sketch of The Proof of Asymptotics of the Estimator

**Theorem 1.** Minimization of the objective function is equivalent to minimizing

$$\frac{1}{n} \sum_t \rho_r(u_{it} - H(x_{it}, \theta)) - \frac{1}{n} \sum_t \rho_r(u_{itt})$$

where $u_{itt} \equiv u_{it} - H(x_{it}, \theta_i(\tau))$, which satisfies

$$Q_{u_{itt}}(\tau|x_{it}) = 0 \quad \text{i.e.,} \quad \tau = \Pr[u_{itt} \leq 0|x_{it}].$$

Denote

$$\overline{H}_t = H(x_{it}, \theta) - H(x_{it}, \theta_i(\tau)),$$

and

$$q_r(u_{it}, x_{it}, \theta) = \rho_r(u_{itt} - \overline{H}_t) - \rho_r(u_{itt}),$$

then

$$\hat{\theta}(\tau) = \arg \min_{\theta \in \Theta} Q_n(\theta) \quad \text{with} \quad Q_n(\theta) = \frac{1}{n} \sum_t q_r(u_{it}, x_{it}, \theta).$$

We first establish consistency by verifying uniform convergence of $Q_n(\theta)$ and the identification condition of $E[Q_n(\theta)]$. Let $B(\theta, \eta)$ be a $\eta$-neighborhood around $\theta$, and let

$$\Delta_{t\eta} = \sup_{\theta \in \Theta} \sup_{\theta' \in B(\theta, \eta)} |q_r(u_{it}, x_{it}, \theta) - q_r(u_{it}, x_{it}, \theta')|,$$
then
\[
\Pr \left( \sup_{\theta \in \Theta} \sup_{\theta' \in B(\theta, \eta)} |Q_n(\theta) - Q_n(\theta') - E(Q_n(\theta) - Q_n(\theta'))| > \varepsilon \right) \\
\leq \Pr \left( \frac{1}{n} \sum_{t=1}^{n} |\Delta t + E \Delta t_n| > \varepsilon \right) \\
\leq \frac{E \left( \frac{1}{n} \sum_{t=1}^{n} [\Delta t + E \Delta t_n] \right)}{\varepsilon} = \frac{2}{\varepsilon} E \Delta t_n
\]

which converges to zero by dominated convergence Theorem.

Next, notice that,
\[
\frac{1}{n} \sum_{t=1}^{n} E \left\{ \rho_t(u_{it\tau} - \bar{H}_t) - \rho_t(u_{it\tau}) | x_{it} \right\} \\
= \frac{1}{n} \sum_{t=1}^{n} \mathbb{1}(\bar{H}_t > 0) E \left\{ \int_{0}^{\bar{H}_t} I(0 \leq u_{it\tau} \leq s) ds | x_{it} \right\} \\
+ \frac{1}{n} \sum_{t=1}^{n} \mathbb{1}(\bar{H}_t < 0) E \left\{ \int_{0}^{\bar{H}_t} I(s \leq u_{it\tau} \leq 0) ds | x_{it} \right\}
\]
under Assumptions A3, we have
\[
E(Q_n(\theta)) \geq \frac{\epsilon_1}{2} E \left[ \bar{H}_t^2 \mathbb{1} \left( \frac{1}{\sigma_t} f_i(F_i^{-1}(\tau)) \geq \epsilon_1 \right) \right],
\]
which, under Assumption A4, is strictly positive. Thus for any \( \varepsilon > 0 \), \( \overline{Q}_n(\theta) \) is bounded away from zero for \( ||\theta - \theta_i(\tau)|| \geq \varepsilon \).

We next derive the limiting distribution of \( \bar{\theta}(\tau) \). Let
\[
\varphi_{it\tau}(\theta) = [\tau - I(u_{it} < H(x_{it}, \theta))] \bar{H}_0(x_{it}, \theta),
\]
and define \( \lambda_{\tau}(\theta) = E \varphi_{it\tau}(\theta) \). Under Assumptions A2 and A3, notice that \( Q_{u_{it}}(\tau | x_{it}) = \)
\( H(x_{it}, \theta_i(\tau)) \), we can show that

\[
\| \lambda_r(\theta) - V_i(\tau)(\theta - \theta_i(\tau)) \| = o(\| \theta - \theta(\tau) \|).
\]

Next, notice that

\[
\sum_t \varphi_{itr}(\hat{\theta}_i(\tau)) = \left\{ \sum_t \left[ \varphi_{itr}(\hat{\theta}_i(\tau)) - \lambda_r(\hat{\theta}_i(\tau)) \right] - \sum_t \left[ \varphi_{itr}(\theta_i(\tau)) - \lambda_r(\theta_i(\tau)) \right] \right\}
\]

\[
+ \sum_t \left[ \varphi_{itr}(\theta_i(\tau)) + \lambda_r(\theta_i(\tau)) \right]
\]

in addition, for \( \theta \) that are close to \( \hat{\theta}_i(\tau) \), we have

\[
\frac{\left\| \sum_t \left[ \varphi_{itr}(\theta_i(\tau)) + \lambda_r(\hat{\theta}_i(\tau)) \right] \right\|}{\sqrt{n} + n \left\| \lambda_r(\hat{\theta}_i(\tau)) \right\|} \leq \sup_{\| \theta - \hat{\theta}_i \| \leq d} \frac{\left\| \sum_t \left[ \varphi_{itr}(\theta) - \lambda_r(\theta) \right] - \sum_t \left[ \varphi_{itr}(\theta_i(\tau)) - \lambda_r(\theta_i(\tau)) \right] \right\|}{\sqrt{n} + n \left\| \lambda_r(\theta) \right\|} + \left\| \frac{1}{\sqrt{n}} \sum_t \varphi_{itr}(\hat{\theta}_i(\tau)) \right\|,
\]

Using similar argument as Huber (1967), we show that the two terms on the right hand side of the above inequality are \( o_p(1) \). Thus under Assumption A5,

\[
\sqrt{n} \left( \hat{\theta}_i(\tau) - \theta_i(\tau) \right) \Rightarrow N(0, \Sigma_i(\tau)).
\]

**Theorem 2** The result of Theorem 2 can be obtained similarly to Xiao and Koenker (2008).
Theorem 3  Notice that \( \hat{\sigma}_t \hat{\beta}_i \hat{\gamma}_i = \sigma_{i,t}(\hat{\beta}_i, \hat{\gamma}_i) = \alpha_i x_{it}(\hat{\alpha}_i) \), and

\[
x_{it}(\hat{\alpha}_i) - x_{it}(\alpha_i) = (1, \hat{\sigma}_{i,t-1}, |u_{i,t-1}|, |u_{i,t-1}|1(u_{i,t-1} < 0))' - (1, \sigma_{i,t-1}, |u_{i,t-1}|, |u_{i,t-1}|1(u_{i,t-1} < 0))' = (0, \hat{\sigma}_{i,t-1} - \sigma_{i,t-1}, 0, \ldots, 0)'
\]

we can show that

\[
\sqrt{n}(\hat{\sigma}_i - \sigma_i) = \mathbf{x}_{it}' \sqrt{n}(\hat{\alpha}_i - \alpha_i) + O_p(n^{-1/2}).
\]

Thus, by results of Theorem 2, let \( \omega_{it} = \mathbf{x}_{it}' \Sigma_{iit} \mathbf{x}_{it} \),

\[
\frac{\sqrt{n}(\hat{\sigma}_i - \sigma_i)}{\omega_{it}} \Rightarrow N(0, 1).
\]
1.10 Appendix C: A Robustness Comparison for Volatility Estimates

We compare the robustness of the maximum likelihood estimates with that of the quantile-regression based estimates under different innovation distributions. First, we generate one thousand time-series of idiosyncratic shocks as defined in 1.5 under different innovation distributions, including normal distribution, t distribution with degrees of freedom 3 and 5 and skewed t distribution with degree of freedom 3 and the skewness parameter 5. Then, we estimate the return volatilities using either the maximum likelihood method with a Gaussian-innovation assumption or using the proposed quantile-regression based method.

<table>
<thead>
<tr>
<th></th>
<th>Normal</th>
<th>$t(df = 5)$</th>
<th>$t(df = 3)$</th>
<th>$sk - t(df = 3, sk = 5)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1. MLE of Conditional Volatility (with a Gaussian-Innovation Assumption)</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$T = 60$</td>
<td>$9.86 \times 10^{-4}$</td>
<td>$8.42 \times 10^{-3}$</td>
<td>$6.69 \times 10^{-3}$</td>
<td>$3.72 \times 10^{-3}$</td>
</tr>
<tr>
<td>$T = 300$</td>
<td>$2.61 \times 10^{-4}$</td>
<td>$4.59$</td>
<td>$5.26 \times 10^{-3}$</td>
<td>$0.123$</td>
</tr>
<tr>
<td>$T = 500$</td>
<td>$1.45 \times 10^{-4}$</td>
<td>$4.98 \times 10^{-3}$</td>
<td>$5.83 \times 10^{-3}$</td>
<td>$0.235$</td>
</tr>
<tr>
<td>2. Quantile-Regression Based Estimates of Conditional Volatility</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$T = 60$</td>
<td>$1.15 \times 10^{-3}$</td>
<td>$1.33 \times 10^{-3}$</td>
<td>$2.27 \times 10^{-3}$</td>
<td>$1.34 \times 10^{-3}$</td>
</tr>
<tr>
<td>$T = 300$</td>
<td>$3.56 \times 10^{-4}$</td>
<td>$5.77 \times 10^{-4}$</td>
<td>$9.82 \times 10^{-4}$</td>
<td>$5.46 \times 10^{-4}$</td>
</tr>
<tr>
<td>$T = 500$</td>
<td>$2.85 \times 10^{-4}$</td>
<td>$4.84 \times 10^{-4}$</td>
<td>$7.12 \times 10^{-4}$</td>
<td>$4.28 \times 10^{-4}$</td>
</tr>
</tbody>
</table>

For different sample sizes $T$ and under different innovation distributions, the upper panel shows the mean square errors (MSE) of the maximum likelihood estimates with a Gaussian-innovation assumption, and the low panel shows the MSE for quantile regression based estimates.

Under correct specification of a Gaussian-innovation distribution, the maximum likelihood estimates of return volatility are very accurate and outperform the proposed quantile-regression based method by having smaller mean-squared errors. However, the small-sample performance of maximum likelihood estimates is problematic under misspecification in innovation distributions. For instance, if the underly-
ing distribution is skewed with fat-tail as in skewed-$t$ distribution, the misspecified Gaussian-innovation assumption would cause severe estimation errors in the maximum likelihood estimates of return volatilities represented by large MSE. While the quantile-regression based method is able to generate more accurate volatility estimates under different return distributions. This result is not surprising at all, since the proposed quantile-regression based method does not relay on innovation distribution assumptions. In conclusion, the proposed quantile-regression based method is able to robustly estimate conditional idiosyncratic volatilities for different return processes across large number of stocks.
1.11 Appendix D: Heteroscedasticity-educed location shift in the regression coefficient

\[ R_{it} = \alpha_{0,t} + \sum_{k=1}^{K} \alpha_{k,t} x_{i,k,t} + \gamma \sigma_{i,t} + \nu_{i,t} \]

If we are concerned that the empirical findings is due to the fact that the variance of \( \nu_{i,t} \), denoted as \( \omega_{i,t}^2 \), is positively correlated with \( \sigma_{i,t} \) and consequently educes a location shift in the coefficient in front of \( \sigma_{i,t} \), \( \gamma \); consider a regression of

\[ R_{it} = \alpha_{0,t} + \sum_{k=1}^{K} \alpha_{k,t} x_{i,k,t} + \gamma \sigma_{i,t} + (a + b\sigma_{i,t})\tilde{\nu}_{i,t}. \]

where \( \tilde{\nu}_{i,t} \) is normalized error term with mean zero and unit variance.

In this case, even the true value of \( \gamma \) is constant over \( \tau \), the quantile of \( R_{it} \), without taking into account of possible heteroscedasticity, the quantile regression estimates of \( \gamma \) will still be a function of \( \tau \) and has the following form

\[ \hat{\gamma} = \gamma + bQ_{\tau}(\tilde{\nu}_{i,t}), \]

where \( Q_{\tau}(\tilde{\nu}_{i,t}) \) is the \( \tau-th \) quantile of the distribution of \( \tilde{\nu}_{i,t} \).

To mitigate the concern of this heteroscedasticity-educed location shift in \( \gamma \), I consider the following reweighed quantile regression:

1. Regress OLS residual \( \tilde{\nu}_{i,t} \) onto \( \sigma_{i,t} \) to consistently estimate \( a \) and \( b \).
2. let \( \tilde{\omega}_{i,t} = \tilde{\sigma} + \tilde{b} \sigma_{i,t} \), and transform the model by dividing both side by \( \tilde{\omega}_{i,t} \), i.e.

\[
\frac{R_{it}}{\tilde{\omega}_{i,t}} = \frac{\alpha_{0,t}}{\tilde{\omega}_{i,t}} + \sum_{k=1}^{K} \frac{\alpha_{k,t}}{\tilde{\omega}_{i,t}} \frac{x_{i,k,t}}{\tilde{\omega}_{i,t}} + \gamma \frac{\sigma_{i,t}}{\tilde{\omega}_{i,t}} + \frac{\nu_{i,t}}{\tilde{\omega}_{i,t}}.
\]
Then, run quantile regression with the transformed variables

\[ R_{it}^* = \alpha_{0,t}^* + \sum_{k=1}^{K} \alpha_{k,t}^* x_{i,k,t}^* + \gamma \sigma_{i,t}^* + v_{i,t}^*, \]

and see if the coefficient \( \gamma \) is still monotonically increasing over \( \tau \).
Chapter 2

Pessimistic Portfolio Selection: An Expected Utility Perspective

Abstract: This paper examines portfolio allocation decision for investors with general pessimistic preferences (GPP) regarding downside risk aversion and out-performing benchmark returns. I show that the expected utility of pessimistic investors can be robustly estimated within a quantile regression framework without assuming asset return distributions. The asymptotic properties of the optimal portfolio weights are derived. Empirically, this method is introduced to construct the optimal fund of CSFB/Tremont hedge-fund indices. Both the in-sample and out-of-sample back-testing results confirm that the optimal mean-GP portfolio outperforms the mean-variance and mean-conditional VaR portfolios.

Keywords: general pessimistic preference, coherent regular risk measures, quantile regression, portfolio allocation, value-at-risk (VaR); conditional value-at-risk (CVaR), backtest
2.1 Introduction

All financial institutions face the same question, that is, how to allocate wealth among alternative assets. Markowitz (1952) proposed that investors should choose portfolio that offers the smallest variance for a given level of expected return. For more than five decades, this mean-variance analysis has served as the standard procedure for portfolios allocation. The mean-variance criterion corresponds to maximizing expected utility if portfolio returns are normally distributed or if investors have quadratic utility, which is fully determined by mean and variance of portfolio returns. However, financial asset returns are generally not normally distributed. Strong empirical evidence against the normality assumption of assets returns has been reported and numerous empirical analysis shows that financial time series tend to be skewed and heavy-tailed (leptokurtic). On the other hand, quadratic utility may not be able to fully capture investors’ different treatment of gains and losses in asset returns. Quadratic preference assumes that investors are as averse to upside gain as they are to downside loss. In practice, however, investors are loss-averse, and care mainly about the loss associated with downside movements and that upside gain should not be penalized. Therefore,

From last decade, researchers and practitioners have shifted from variance to downside-risk measures in portfolio and financial risk management. The leading examples are value-at-risk (VaR) and conditional value-at-risk (CVaR, Rockafellar and Uryasev, 2000)\(^1\). The latter is also called the expected shortfall or expected tail loss. VaR corresponds to the market value of potential loss that is exceeded with a certain probability over a given time period. As a intuitive measure of risk, VaR has become an industry standard for measuring risk and has gained considerable popularity in the application of assets allocation. However, as showed by Artzner, Delbaen, Eber

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\(^1\)VaR has now been incorporated in the requirements of the Basel II Capital Accord the Basel II capital accord; while CVaR is used in the Solvency II framework for risk measurement.
and Heath (1999), VaR measures are not coherent due to the lack of sub-additivity; therefore minimizing portfolio VaR might fail to induce diversification.

The most prominent coherent risk measure is conditional value-at-risk (CVaR, also called the expected shortfall or expected tail loss). As a natural alternative for VaR, CVaR is defined as the expected loss given losses exceed VaR for a certain confidence level. CVaR offers a sound picture of downside risk and has desirable properties of a coherent risk measure, i.e., monotonicity, subadditivity, translation invariance, and positive homogeneity. Kusuoka (2001) further imposed two additional regularity conditions on coherent risk measures, law invariance and comonotonic additivity, to construct a coherent regular risk (CRR) measures. Furthermore, Bassett, Koenker and Kordas (2005) show that a coherent regular risk measure must be a pessimistic measure that leads to a pessimistic decision criterion for asset allocation. In addition, Melenberg and Polbennikov (2005 a,b), among others, study the implementation of coherent regular risk measures in portfolio selection, and show that the mean-variance and mean-CRR analyses produce statistically similar portfolios of common equities.

However, due to the lack of utility motivation, investors often find themselves facing such a dilemma: on one hand, literature emphasizes the superior of coherent regular risk measures and promote the implementation of coherent regular risk measures in assets allocation; while on the other hand, there is still no clear guideline of which investors would benefit from shifting from the conventional variance measures to the pessimistic measures. It is clear that investors with quadratic utility, which is fully determined by the first and second moments of wealth level, should allocate capital according to the mean-variance analysis to maximize their expected utility. Then, a natural question to ask is what type of investors should adopt CVaR or other coherent risk measures to maximize their expected utility.

This paper takes utility considerations seriously and uses the utility specification as a guide for choosing market risk measures and solving portfolio allocation problem. I first point out that the mean-CVaR analysis, which minimizes portfolio’s
conditional value-at-risk for a given level of expected return, fails to fully capture investors’ attitude towards losses and gains. I show that the mean-CVaR analysis would imply a truncated utility function, which focuses exclusively on the extreme downside risk while completely ignores portfolio upside gains. Therefore, the mean-CVaR analysis fails to maximize most investors’ expected utility.

Second, I propose a piece-wise linear concave utility function with kinks determined by endogenous reference points to capture investors’ pessimistic preferences and attitude towards losses and gains. By maximizing the expected utility of investors with general pessimistic preferences, I obtain a coherent risk measure, which essentially is a weighted summation of portfolio CVaRs at different quantiles. The weights and quantiles are fully determined by investors’ preferences. Therefore, investors would be able to tailor this general pessimistic risk measures to better reflect their risk-tolerance across the whole portfolio return distribution and incorporate different benchmarks. As showed by Bassett, Koenker and Kordas (2005), the utility maximization problem for investors with general pessimistic preferences can be solved within a quantile regression framework without imposing distribution assumptions on asset returns. Furthermore, the statistical properties of the optimal portfolio weights for investors with general pessimistic preferences are studied.

Third, I apply the mean-GP criterion to construct the optimal fund of CSFB/Tremont hedge-fund indices. The departure from the normal distribution of asset returns is even more accentuated in the hedge fund environment, which causes the mean-variance analysis less applicable. Moreover, neither a quadratic utility imposed by mean-variance analysis nor a upper truncated utility imposed by the mean-CVaR analysis is capable of fully reflecting the unconventional preferences of managers of (funds of) hedge funds induced by hedge funds’ distractive fee structures. Therefore, the mean-GP criterion studies in this paper is capable of maximizing investors’ expected utility in the presence of non-Gaussian returns and better capturing investors full-range preferences over all possible portfolio returns. Both the in-sample and out-
of-sample backtests confirm that the optimal mean-CPP portfolio outperforms both the mean-variance and mean-CVaR portfolios by generating higher returns.

The rest of this paper proceeds as follows. Section 2 describes the pessimistic risk measures and set up the utility framework to further motivate the mean-CPP criterion in portfolio allocation. Section 3 derives the asymptotic properties of the optimal portfolio weights yielded by the mean-CPP analysis. Section 4 introduces the mean-CPP analysis to construct optimal fund of CSFB/Tremont hedge-fund indices. Section 5 concludes.
2.2 Pessimistic Risk Measures

For a long time, the variance of portfolio return has been the predominant market risk measure. Markowitz (1952) proposed that investors should construct a portfolio that offers the smallest return variance for a given level of expected return. However, one definite drawback of variance as a risk measure is that it may not properly reflect the whole return distributions and it increases with both positive and negative deviations from the mean. Therefore, from last decade, researchers and practitioners have shifted from the variance measure to the downside-risk measures in financial risk management and portfolio allocations since investors mainly concern about the risk exposure from the extreme lower-tail events. The most famous examples are VaR and CVaR. And the popularity of such downside risk measures has grown considerably due to their intuitive notions and the regulatory need for establishing risk measurement and management systems for financial institutions. VaR is defined as the market value of a potential loss that is exceeded at a given confidence level \( \alpha \). For instance, the 95\% VaR is an estimate of loss which is exceeded with 5\% probability. From a statistical point of view, for a given level of \( \alpha \), VaR corresponds to the \( \alpha \)-th quantile of the potential loss distribution of the portfolio. The CVaR measure is closely related to VaR and is defined as the expected loss given the losses exceed VaR.

Despite of its popularity, VaR as a risk measure has been criticized by financial applicants. An important criticism to VaR is that VaR is not a coherent risk measure due to lack of sub-additivity (e.g. Wu and Xiao, 2002). Following the axiomatic approach, Artzner et al. (1999) define a risk measure as coherent if it satisfies the following four axioms.

**Definition 1** (Artzner et al) A mapping \( \rho = \rho_0 : \chi \rightarrow \mathbb{R} \cup \{+\infty\} \) is called a coherent measure of risk if it satisfies the following conditions for all \( X, Y \in \chi \).

- **Monotonicity:** if \( X \leq Y \), then \( \rho(X) \geq \rho(Y) \).


Translation Invariance: if $a \in \mathbb{R}$, then $\rho(X + a) = \rho(X) - a$.

Positive Homogeneity: if $\lambda \geq 0$, then $\rho(\lambda X) = \lambda \rho(X)$.

Sub-additivity: $\rho(X + Y) \leq \rho(X) + \rho(Y)$.

The four axioms have clear financial sense and departure from them is always leads to paradox. Monotonicity says that if one position always lose at least as much as the other in all states, then it should have a risk measure which is at least as large as the other; a risk measure satisfies the translation invariance axiom if adding an amount $a$ of risk-free asset to a position reduces the capital requirement by the same amount; positive homogeneity says that the risk exposure of a position grows in a linear way as the size of the position increases; Sub-additivity specifies that the risk of a portfolio never exceeds the sum of the risks of its component assets, which closely related to the concept of reducing risk by diversification in portfolio allocation.

According to the definition of coherent risk measures, variance is ruled out as a coherent risk measure due to the lack of monotonicity since the increase in portfolio return fails to decrease the measure of return variance. VaR as a risk measure fails to satisfy the property of sub-additivity. That means VaR of a portfolio may be greater than the sum of individual VaRs of its component assets, and consequently, minimizing portfolio VaR may fail to stimulate diversification. In addition, the VaR based risk management is criticized for merely focusing on controlling for the probability of loss, rather than its magnitude (Basak and Shapiro [1999]).

While VaR doesn’t fall into the category of coherent risk measurement as not being sub-additive; CVaR has been suggested as a remedy for VaR based downside risk measure. CVaR is defined as the expected loss exceeding VaR. More specifically, The CVaR at $\alpha$ level is the expected loss of portfolio value given that a loss is occurring at or below the $\alpha$-quantile, denoted as $q_\alpha$:

$$CVaR_\alpha = -E(x|x < q_\alpha).$$
Unlike VaR, which is insensitive to the magnitude of loss beyond a certain percentile of portfolio returns, CVaR weights large losses by the probability of their occurrences. CVaR is a coherent measure of financial risk and has been intensively researched in various portfolio allocation applications. See Pflug (2000); Rockafellar and Uryasev (2001), Acerbi et al. (2001), Acerbi and Tasche (2001) for related discussions.

Furthermore, kusuoka (2001) imposed two additional regularity restrictions on the set of coherent risk measures:

- i) $\rho$ is law invariant, that is, $\rho(X) = \rho(Y)$ if $X, Y \in \chi$ have the same probability law.

- ii) $\rho$ is comonotonically additive, that is, $X, Y \in \chi$ comonotone implies that $\rho(X + Y) = \rho(X) + \rho(Y)$.

where comonotonicity is defined as:

**Definition 2** Two random variables $X, Y \in \chi$ are comonotonic if there exists a third random variable $Z \in \chi$ and increasing functions $f$ and $g$ such that $X = f(Z)$ and $Y = g(Z)$.

That is, two random variables are comonotonic if they are monotonic transformations of the same random variable. These two additional axioms also have clear meaning. The law invariance requires that two assets have the same risk if they have the same distribution function. While the second axiom states that subadditivity becomes additivity if two assets are comonotonic.

Bassett, Koenker and Kordas (2005) show that a coherent regular risk measure must be a *pessimistic* measure, and vice versa. Intuitively, a pessimistic risk measure assigns higher weights to the likelihood of the most unfavorable outcomes and discounts that of the most favorable outcomes, and is defined as following.
**Definition 3** (Bassett et al, 2005) A risk measure \( \rho \) is **pessimistic** if, for any random variable \( X \) with cumulative distribution function \( F_X(\cdot) \) and some probability measure \( \varphi \) on \([0, 1]\),

\[
\rho(X) = \int_0^1 \rho_{\nu_\alpha}(X)d\varphi(\alpha),
\]

where \( \rho_{\nu_\alpha} \) is the expected shortfall of \( X \) at the \( \alpha \)-th level

\[
\rho_{\nu_\alpha}(X) = -\int_0^\alpha F_X^{-1}(t)dt.
\]

I therefore call a coherent regular risk measure as a pessimistic risk measure. Since CVaR enjoys all the nice properties of being a pessimistic risk measure, recently, researchers and practitioners advocate the mean-CVaR analysis for portfolio selection, i.e., minimizing portfolio CVaR at a given level of expected portfolio return.

Let \( q_\alpha \) be the \( \alpha \)-th quantile of the return distribution, the mean-CVaR approach corresponds to maximize a simple truncated utility function of investors:

\[
u(R) = \begin{cases} R/\alpha, & \text{if } R \leq q_\alpha \\ 0, & \text{otherwise} \end{cases}
\]

(2.1)

where the 100\( \alpha \)% least favorable outcomes are the major concern and the rest \( 1 - \alpha \) proportion of all possible outcomes, including those most favorable ones, are discarded entirely. In the mean-CVaR setting, investors’ expected utility is the negative value of CVaR of the portfolio return distribution corresponding to \( \alpha \)-th quantile:

\[
E[u(R)] = \int_{-\infty}^{+\infty} u(R)dF(R) = \frac{1}{\alpha} \int_{-\infty}^{q_\alpha} RdF(R) = -CVaR_\alpha.
\]

CVaR evaluates the value (or risk) of an investment in a conservative way, focusing on the less profitable outcomes. For high values of \( \alpha \) it ignores the most profitable but unlikely possibilities, for small values of \( \alpha \) it focuses on the worst losses. The above utility function (2.1) reflects a strong motivation of downside risk aversion; however, it oversimplifies by assuming investors become indifferent to the change of portfolio return after a benchmark. By putting all the weights on high losses, such a
utility function, so does mean-CVaR analysis fails to balance risk and profit in assets allocation.
2.3 General Pessimistic Portfolio Allocation

The CVaR measure corresponds to the simplest case of pessimistic preference, and the mean-CVaR criterion maximizes the expected utility for investors with a highly unrealistic truncated utility function. Consequently, the mean-CVaR criterion may fail to maximize the expected utility for investors with sensible preferences.

This paper takes utility considerations seriously and uses the utility specification as a guide for choosing market risk measurement and solving portfolio allocation problem. A very general utility function capturing the general pessimistic preference which involves the out-performance of benchmarks and loss aversion is the following piecewise linear utility with diminishing marginal utility

\[
U(W) = \begin{cases} 
  a_{M+1}W + b_{M+1}, & \text{if } W \in (\widehat{W}_M, +\infty) \\
  \vdots & \\
  a_i W + b_i, & \text{if } W \in (\widehat{W}_{i-1}, \widehat{W}_i] \\
  \vdots & \\
  a_1 W + b_1, & \text{if } W \in (-\infty, \widehat{W}_1] 
\end{cases}
\]

where \( M \) is the number of kinks in utility function, and \( \{\widehat{W}_i\}_{i=1}^M \) are \( M \) reference points of wealth, and \( a_1 > a_2 > \cdots > a_{M+1} \).

The slope of the \( i \)-th line segment, \( a_i \), indicates the degree of loss aversion. For a loss-averse investor, I would expect that the utility function becomes flatter given increases in wealth, which also implies diminishing marginal utility as the level of wealth increases. As a result, I assume that \( a_{i-1} \geq a_i \), for \( \forall i \in [1, M+1] \).

Given the initial wealth \( W_0 \), in a single period, the utility function of wealth becomes a function of the portfolio return rate \( R \) (Kahnemann & Tversky (1979)
have found that people focus on returns more than wealth levels). Thus I consider

\[ U(W) = u(R) = \begin{cases} 
    a_{M+1}R + b_{M+1}, & \text{if } R \in (q_M, +\infty) \\
    \vdots \\
    a_iR + b_i, & \text{if } R \in (q_{i-1}, q_i] \\
    \vdots \\
    a_1R + b_1, & \text{if } R \in (-\infty, q_1] 
\end{cases} \quad (2.2) \]

where \( q_i = q_{\alpha_i} = F_{R}^{-1}(\alpha_i) \) corresponds to the \( \alpha_i \)-th quantile of return distribution, thus the \( i \)-th reference point equals to the \( \alpha_i \)-th quantile of the portfolio’s return distribution. I assume that \( \alpha_0 \) equal 0 and \( \alpha_{M+1} \) equal 1, so the quantiles of \( \alpha_0 \) and \( \alpha_{M+1} \) correspond to the negative and positive infinity, respectively.

In this paper, I consider portfolio selection based on the above general utility function. Suppose the portfolio return \( R \) has a distribution function \( F(\cdot) \). For convenience of analysis, I use the same notation as Bassett, Koenker and Kordas (2006) and define the \( \alpha \)-risk of \( R \) as

\[ \varrho_\alpha(R) = -\frac{1}{\alpha} \int_{-\infty}^{q_\alpha} RdF(R), \]

where \( q_\alpha \) is the \( \alpha \)-th quantile of the return distribution. Denoting the mean of distribution of portfolio return \( R \) as \( \mu_R \), the expected utility is then given by

\[ E[u(R)] = a_{M+1}\mu_R - \sum_{i=1}^{M} (a_i - a_{i+1})\alpha_i\varrho_{\alpha_i}(R) - \sum_{i=1}^{M} (b_{i+1} - b_i)\alpha_i - b_{M+1}. \quad (2.3) \]

Thus, maximizing expected utility under general loss aversion is equivalent to minimization of a weighted summation of \( \alpha \)-risks \( \sum_{i=1}^{M} \lambda_i\alpha_i\varrho_{\alpha_i}(R) \), where the weights are determined by investor’s preference over risk and profit—the slopes of line segments.
of the utility function (the marginal utilities).

I consider an investment decision over $L$ underlying assets with random returns $r = (r_1, ..., r_L)'$ that maximizes the expected utility. If I construct a portfolio by choosing portfolio weights $\omega = (\omega_1, ..., \omega_L)'$, the portfolio return rate $R$ is equal to $\omega'r$. For identification purpose, I standardize the weights so that they satisfy the constraint $\sum_{i=1}^{L} \omega_i = 1$.

The optimal portfolio choice for an investor with utility function $u(R)$ corresponds to the following maximization problem

$$(P1) \begin{cases} \max_{\omega} & E[u(\omega'r)] \\ \text{s.t.} & \omega'l = 1. \end{cases}$$

where $l = (1, \cdots, 1)'$. My purpose is to find the optimal weights $\omega = (\omega_1, ..., \omega_L)$ and the portfolio return distribution, which is a function of portfolio weight $\omega$.

Let $\lambda_i = A_i - A_{i-1}$, then by (2.3), the portfolio selection problem (P1) can be equivalently written as

$$(P1') \begin{cases} \max_{\omega} & \left[\sum_{i=1}^{M} \lambda_i \alpha_i \varrho_{\alpha_i}(\omega'r) - a_{M+1}\mu_R + \sum_{i=1}^{M} (b_{i+1} - b_i)\alpha_i \right] \\ \text{s.t.} & \omega'l = 1. \end{cases}$$

Notice that the value of $\alpha$-risk can be obtained from the following optimization problem (Bassett, Koenker and Kordas, 2005):

$$\varrho_{\alpha}(X) = \frac{1}{\alpha} \min_{\xi} E\rho_{\alpha}(X - \xi) - \mu_X,$$

where $\rho_{\alpha}()$ is the check function defined as $\rho_{\alpha}(x) = x(\alpha - I(x < 0))$, the utility maximization problem (P1) can be further written as a quantile regression problem.

I summarize the above result in Proposition 1.

**Proposition 1.** The optimal portfolio selection based on utility function (2.2)
is determined by the following quantile regression problem:

\[ \min_{\omega, \xi_i, i = 1, \ldots, M} \left[ \sum_{i=1}^{M} \lambda_i E\rho_{\alpha_i}(\omega'r - \xi_i) - \left( \sum_{i=1}^{M} \lambda_i \alpha_i + a_{M+1} \right) \mu_{\omega'r} \right]. \]

In practice, the return distribution is unknown and thus \( E\rho_{\alpha_i}(\omega'r - \xi_i) \) is unknown. If there is a sample of \( N \) observations from the return distribution of assets return, say \( r_t, t = 1, \cdots, N \), replacing the expectations by their sample analogues, the optimal portfolio weights can be determined by solving the following minimization problem:

\[ \min_{\omega, \xi_i, i = 1, \ldots, M} \left[ \sum_{i=1}^{M} \lambda_i \sum_{t=1}^{N} \rho_{\alpha_i}(\omega'r_t - \xi_i) - \left( \sum_{i=1}^{M} \lambda_i \alpha_i + a_{M+1} \right) \sum_{t=1}^{N} \omega'r_t \right]. \] \hspace{2cm} (2.4)

Solution of the above optimization problem gives optimal weights of portfolio that maximizes utility function (2.2).

The approach developed here for optimal portfolio selection differs from other approaches, especially, mean-variance analysis, in several important respects. A prerequisite to use of the Markowitz framework is either that the relevant distribution of asset returns be normally distributed or that utility is only a function of the first two moments. However, it’s well known that many financial returns are not normally distributed, and mean-variance efficient portfolio will not be optimal when an investor’s utility can not be fully captured by mean and variance.

In this model, the weighted summation of conditional VaRs, called the general pessimistic risk measurement, serves as a coherent regular risk measure and reflects investor preferences over the whole portfolio return distribution.
2.4 Statistical Properties of the Optimal Portfolio Weights

The portfolio selection problem corresponds to the following quantile regression estimation problem:

\[
\begin{align*}
\min_{\omega, \xi, i=1, \ldots, M} & \quad \sum_{i=1}^{M} \lambda_i \sum_{t} \rho_{\alpha_i} (\omega' r_t - \xi_i) \\
\text{s.t.} & \quad \sum_{j=1}^{L} \omega_j = 1, \omega_j \geq 0.
\end{align*}
\]

where \( \omega \) is the vector of weights, i.e.

\[\omega = (\omega_1, \ldots, \omega_L)',\]

and standardized by \( \sum_{j=1}^{L} \omega_j = 1.\)

To study the above quantile regression problem and incorporate the restriction \( \sum_{j=1}^{L} \omega_j = 1, \) I may transform the data

\[y_t = r_{1t}, \quad x_t = (r_{1t} - r_{2t}, \ldots, r_{1t} - r_{Lt})^T, \quad \beta = (\beta_2, \ldots, \beta_L)^T,\]

then the problem can be re-written as the following unrestricted quantile regression

\[
\min_{\beta, \xi, i=1, \ldots, M} \left\{ \sum_{i=1}^{M} \lambda_i \sum_{t} \rho_{\alpha_i} (y_t - \xi_i - \beta^T x_t) \right\}.
\]

Corresponding the original quantile regression, I have \( \omega = \left( 1 - \sum_{j=2}^{L} \beta_j, \beta_2, \ldots, \beta_L \right), \) and the estimated weights are given by

\[\hat{\omega} = \left( 1 - \sum_{j=2}^{L} \hat{\beta}_j, \hat{\beta}_2, \ldots, \hat{\beta}_L \right).\]
Let $r_{jt}$ be the demeaned returns, i.e.

$$r_{jt} = r_{jt} - N^{-1} \sum_{t=1}^{N} r_{jt}$$

and define

$$D_n = \begin{bmatrix}
\frac{1}{n} \sum_{t=1}^{n} (r_{1t} - \bar{r})^2 & \cdots & \frac{1}{n} \sum_{t=1}^{n} (r_{1t} - \bar{r}) (r_{Lt} - \bar{r}) \\
\vdots & \ddots & \vdots \\
\frac{1}{n} \sum_{t=1}^{n} (r_{Lt} - \bar{r}) (r_{Lt} - \bar{r}) & \cdots & \frac{1}{n} \sum_{t=1}^{n} (r_{Lt} - \bar{r})^2
\end{bmatrix}$$

and

$$D = \begin{bmatrix}
\mathbb{E}(r_{1t} - \bar{r})^2 & \cdots & \mathbb{E}(r_{1t} - \bar{r}) (r_{Lt} - \bar{r}) \\
\vdots & \ddots & \vdots \\
\mathbb{E}(r_{Lt} - \bar{r}) (r_{Lt} - \bar{r}) & \cdots & \mathbb{E}(r_{Lt} - \bar{r})^2
\end{bmatrix}.$$  

The asymptotic behavior of the optimal weights are summarized in the following Proposition.

**Proposition 2:** Under regularity conditions specified in the Appendix, the asymptotic distribution of the optimal weights is given by

$$\sqrt{n} (\hat{\omega} - \omega) \Rightarrow N (0, \Sigma_{\omega})$$

where

$$\Sigma_{\omega} = \frac{\sum_{i=1}^{M} \sum_{j=1}^{M} \lambda_i \lambda_j (\min(\alpha_i, \alpha_j) - \alpha_i \alpha_j)}{\sum_{i=1}^{M} \sum_{j=1}^{M} \lambda_i \lambda_j f (F^{-1}(\alpha_i)) f (F^{-1}(\alpha_j))} \begin{bmatrix}
1^\top D^{-1} 1 & -1^\top D^{-1} \\
-D^{-1} 1 & D^{-1}
\end{bmatrix},$$

and $1$ is a vector of 1’s.

In some applications, the investor’s decision is subject to some further restric-
tions on returns, and I need to consider the optimization problem with restriction to constant returns, i.e.

\[ \omega' \sum_t r_t = c. \]

In this case, the portfolio optimization problem becomes

\[
\begin{align*}
\min_{\omega, \xi_{i}, i=1, \ldots, M} \quad & \sum_{i=1}^{M} \lambda_i \sum_t \rho_{\alpha_i} (\omega' r_t - \xi_i) \\
\text{s.t.} \quad & \sum_{j=1}^{L} \omega_j = 1, \omega_j \geq 0. \\
& \omega' \sum_t r_t = c.
\end{align*}
\]

Under the restrictions \( \sum_{j=1}^{L} \omega_j = 1, \) and \( \omega' \sum_t r_t = c, \) denote

\[
c_{n2} = \frac{c - \sum_t r_{1t}}{\sum_{t=1}^{N} (r_{2t} - r_{1t})}, c_{nj} = \frac{\sum_{t=1}^{N} (r_{jt} - r_{1t})}{\sum_{t=1}^{N} (r_{2t} - r_{1t})}, \quad j = 3, \ldots, L
\]

then I may transform the original data to

\[
y_t^* = r_{1t} + c_{n2}(r_{2t} - r_{1t}), \\
x_t^* = (x_{3t}^*, \ldots, x_{Lt}^*)', \\
x_{jt}^* = [(r_{jt} - r_{1t}) - c_{nj}(r_{2t} - r_{1t})], \quad j = 3, \ldots, L
\]

The original quantile regression with return restriction can now be re-written as the following unrestricted quantile regression

\[
\hat{\beta}_* = \arg \min_{\beta, \xi_{i}, i=1, \ldots, M} \sum_{i=1}^{M} \sum_t \rho_{\alpha_i} (y_t^* - \xi_i - \beta^T x_t^*)
\]
where
\[
\beta = (\beta_3, \ldots, \beta_L)^\top, \quad \hat{\beta} = (\hat{\beta}_3, \ldots, \hat{\beta}_L)^\top
\]
\[
x_t^* = [(r_{3t} - r_{1t}) - c_{n3}(r_{2t} - r_{1t})], \ldots, [(r_{Lt} - r_{1t}) - c_{nL}(r_{2t} - r_{1t})]^\top.
\]
Corresponding to the weights in original assets,
\[
\omega = \left(1 - c_{n2} + \sum_{j=3}^L (c_{nj} - 1)\beta_j, c_{n2} - c_{n3}\beta_3 - \cdots - c_{nL}\beta_L, \beta_3, \ldots, \beta_L\right),
\]
which can be estimated by
\[
\hat{\omega}^* = \left(1 - c_{n2} + \sum_{j=3}^L (c_{nj} - 1)\hat{\beta}_j, c_{n2} - c_{n3}\hat{\beta}_3 - \cdots - c_{nL}\hat{\beta}_L, \hat{\beta}_3, \ldots, \hat{\beta}_L\right)^\top.
\]
Corresponding to \(x_t^*\), I define \(D^* = \lim \frac{1}{n} \sum_{t=1}^n (x_t^* - \bar{x}^*)^\top (x_t^* - \bar{x}^*)^\top\) and assume that \(D^*\) is positive definite.

I summarize the asymptotics of the restricted estimator in the following Theorem.

**Proposition 3:** Under regularity conditions, the asymptotic distribution of the estimated weights for the portfolio under return restriction is given by
\[
\sqrt{n} [\hat{\omega}^* - \omega] \Rightarrow N(0, \Sigma_T)
\]
where
\[
\Sigma_T = \frac{\sum_{i=1}^M \sum_{j=1}^M \lambda_i \lambda_j (\min(\alpha_i, \alpha_j) - \alpha_i \alpha_j)}{\sum_{i=1}^M \sum_{j=1}^M \lambda_i \lambda_j f(F^{-1}(\alpha_i)) f(F^{-1}(\alpha_j))} \begin{bmatrix}
\Upsilon_1^\top D^{*-1} \Upsilon_1 & \Upsilon_1^\top D^{*-1} \Upsilon_2 & \Upsilon_1^\top D^{*-1} \\
\Upsilon_2^\top D^{*-1} \Upsilon_1 & \Upsilon_2^\top D^{*-1} \Upsilon_2 & \Upsilon_2^\top D^{*-1} \\
D^{*-1} \Upsilon_1 & D^{*-1} \Upsilon_2 & D^{*-1}
\end{bmatrix}
\]
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\[ \gamma_1^\top = \begin{bmatrix} c_{n3} - 1 & \cdots & c_{nL} - 1 \end{bmatrix}, \]
\[ \gamma_2^\top = \begin{bmatrix} -c_{n3} & \cdots & -c_{nL} \end{bmatrix}. \]
2.5 Empirical Application

In this section, I apply the proposed portfolio selection method to financial data and compare it with the widely-used mean-variance and mean-CVaR portfolio selection methods. In particular, I consider the general pessimistic (GP) portfolio allocation problem for a fund of hedge funds (FoF) to allocate capital across different hedge fund strategies, and illustrate the flexibility and the improvement of the mean-GP criterion over the conventional mean-variance and mean-CVaR methods.

2.51 General Pessimistic Portfolio Allocation for Fund of Funds

A fund of hedge funds (FoF) is a hedge fund that allocates its capital to many different hedge funds. A FoF allows the investor to conservatively incorporate hedge funds into a traditional portfolio and creates an optimal portfolio that diversifies across different hedge fund strategies to generate stable returns under most market conditions.

The return series of hedge funds typically exhibit non-normal properties with non-zero skewness and high excess kurtosis (e.g. Amin and Kat, 2003; Lo, 2001). In addition, the payoff schemes of hedge funds elicit asymmetric treatment of upside gain and downside risk. Specifically, hedge funds general specify a hurdle rate, and the performance fee is only collected on profits if the annualized performance exceeds this benchmark rate irrespective of the market condition. Therefore, except of controlling downside risk and maximizing upside gain, breaching a benchmark rate is crucial for hedge fund performance. As a result, mean-variance portfolio analysis is inadequate in dealing with portfolios involving hedge funds for ignoring these higher moments of return distribution. It also penalizes the upside deviations from the mean return and while underestimating downside risk. Therefore, several alternative approaches have been developed in the hedge fund portfolio literature to incorporate investors’
preferences for higher-order moments into constructing optimal FoF. Davies et al. (2004), Berenyi (2001 and 2005), among others, have considered higher moments of asset’s return distribution to determine the set of the mean-variance-skewness-kurtosis effect fund of hedge fund. However, given the limited historical data, it is difficult to accurately estimate high-dimension moment matrices, e.g. skewness-coskewness matrix (3D) and kurtosis-cokurtosis (4D) matrix. In addition, the optimal portfolio obtained in the four-moment space can not guarantee to be a global optimal solution since the mean-variance-skewness-kurtosis efficient frontier is a non-convex surface. On the other hand, while focusing exclusively on the downside, the mean-VaR/CVaR approaches fail to be precisely related to investors’ expected utility and fail to motivate upside gains in portfolio returns.

Compared with previous methods in solving FoF’s optimal asset allocation problem, the mean-GP criterion incorporates investors' asymmetric treatment to both downside risk and upside gain. The mean-GP method, derived in a quantile-regression framework, is also able to robustly estimate investors’ expected utility in the presence of assets with non-normal return distributions. Given the fact that FoF indeed are quite heterogeneous group marked by their own distinctive proprietary strategies, this method allows investors to tailor parameters used to weight different risk measures to better address the special issues of hedge fund allocation.

2.52 Data and Descriptive Statistics

In this paper, following the most existing research on FoF allocation (e.g. Amenc and MArtellini (2002), Lamm (2003), Agarwal and Naik (2004), Morton et al. (2006)), I use hedge funds indexes to represent the different classes of hedge fund strategies. In particular, I focus on four main CSFB/Tremont hedge fund indexes, namely the index of Event Driven (ED), Convertible Arbitrage (CA), Long/Short Equity (LS) and Dedicated Short Bias (DS), which are classified by their types of hedge fund strategies
Table 2.1: Descriptive Statistics for Four CSFB/Tremont Hedge Fund Indexes
The table shows the means, variance, skewness, kurtosis, minimum, maximum and different quantiles of returns for four CSFB/Tremont hedge funds indexes, Event Driven (ED), Convertible Arbitrage (CA), Long/Short Equity (LS) and Dedicated Short Bias (DS), during January 1994 to December 2004.

<table>
<thead>
<tr>
<th>Strategy</th>
<th>Mean</th>
<th>Var</th>
<th>Skew</th>
<th>Kurto</th>
<th>Min</th>
<th>max</th>
<th>Q1%</th>
<th>Q99%</th>
</tr>
</thead>
<tbody>
<tr>
<td>E-D</td>
<td>0.95</td>
<td>2.54</td>
<td>-3.26</td>
<td>25.93</td>
<td>-11.77</td>
<td>3.68</td>
<td>-3.1</td>
<td>3.58</td>
</tr>
<tr>
<td>C-A</td>
<td>0.71</td>
<td>1.76</td>
<td>-1.29</td>
<td>5.98</td>
<td>-4.68</td>
<td>3.57</td>
<td>-4.64</td>
<td>3.46</td>
</tr>
<tr>
<td>L/S</td>
<td>1.0</td>
<td>8.02</td>
<td>0.20</td>
<td>7.02</td>
<td>-11.43</td>
<td>13.01</td>
<td>-7.46</td>
<td>11.14</td>
</tr>
<tr>
<td>S-B</td>
<td>-0.04</td>
<td>23.52</td>
<td>0.84</td>
<td>5.0</td>
<td>-8.69</td>
<td>22.71</td>
<td>-8.65</td>
<td>13.76</td>
</tr>
</tbody>
</table>

and consistent of 56.2% of market share in terms of assets under management (AUM) based on January 2008 Credit Suisse/TREMONT Hedge Fund Index Factsheet. My sample includes 168 monthly returns of four CSFB/Tremont indexes from January 1994 to December 2007.

One concern regarding working with hedge fund indexes is that the index for different hedge fund strategy normally contains different number of funds. Amin and Kat (2002) and Davies, Kat and Lu (2003) show that the properties of hedge fund portfolio return processes vary substantially with the number of funds in the portfolio. As a result, different hedge fund indexes may behavior quite differently not only because they follow different strategies, but also due to the different number of funds used to calculate the index. However, given the fact that hedge fund indexes have been well known and severed as the basis for investment decision, and the limitation of other methods in constructing representative hedge fund portfolios, I accede most existing empirical studies and focus on the optimal FoF allocation using the well-recognized hedge fund indexes.

Table 1 reports the descriptive statistics for the monthly returns on the four Credit Suisse/TREMONT Hedge Fund Indexes during the sample period. The monthly return process for dedicated short bias strategy exhibits a low average return, relatively large variance as a trade-off for the potential of upside gain.
I also show the correlation matrix between the returns on different hedge fund
indexes in Table 2.

<table>
<thead>
<tr>
<th></th>
<th>E-D</th>
<th>C-A</th>
<th>L/S</th>
<th>S-B</th>
</tr>
</thead>
<tbody>
<tr>
<td>E-D</td>
<td>1</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>C-A</td>
<td>0.58</td>
<td>1</td>
<td></td>
<td></td>
</tr>
<tr>
<td>L/S</td>
<td>0.67</td>
<td>0.31</td>
<td>1</td>
<td></td>
</tr>
<tr>
<td>S-B</td>
<td>-0.63</td>
<td>-0.28</td>
<td>-0.72</td>
<td>1</td>
</tr>
</tbody>
</table>

Table 2: Correlation Matrix Between Four CFSB/Tremont Hedge Fund Indexes

Table 2 indicates positive correlations between three hedge fund strategies, namely event driven, convertible arbitrage and long/short equity; while, the return of
dedicated short bias strategy is negatively correlated with the return of other three
strategies since the strategy takes more short positions than long positions and earn
returns by maintaining net short exposure in its equity positions.

2.53 Design of Investment Strategies

I compare investors using the mean-variance approach, the mean-CVaR approach,
and investor that uses the proposed mean-GP criterion based on a linear spline utility
function representing a pessimistic preference. In particular, for the second and third
type investors, I consider two cases in as described in (2.2):

- \textit{mean – CVaR}: \( A_1 = 1, A_2 = 0 \) and \( \alpha = 1\% \),
- \textit{mean – GP}: \( A_1 = 1, A_2 = 0.5, A_3 = 0.3 \) and \( \alpha_1 = 5\%, \alpha_2 = 50\% \).
The different preference parameters are chosen to illustrate the trade-off between the upside gain and the downside risk, to show the limitation of mean-variance and mean-CVaR methods in improving the overall portfolio performance, and to see how allocations change if investor becomes relatively more pessimistic.

2.54 Empirical Results

I now solve the optimal hedge fund allocation problem using the four main CSFB/Tremont hedge fund indexes for mean-variance investor, mean-CVaR investor and investor that selects her portfolio using the proposed approach. The optimal hedge fund allocation for investor with general pessimistic preference is solved as defined in eq (2.4).

As argued in the previous discussion, The mean-variance framework, focusing exclusively on the first two return moments, may fail to fully capture the complexity of hedge fund return process; while, the mean-CVaR, implicitly assuming investor with a highly unrealistic utility function, may fail to pursue upside gain. I expect that the third investor would outperform the other two conventional investors. The empirical analysis confirms my conjecture. The mean-GP optimal portfolio is able to generate a higher average portfolio return with a more positively skewed return distribution. Table 3 reports the optimal portfolio weights obtained by mean-variance, mean-CVaR and mean-GPP methods. It also presents the mean, variance, skewness, kurtosis, quantiles and conditional shortfalls/windfalls for the portfolio return process generated by different methods.
<table>
<thead>
<tr>
<th></th>
<th>$\omega_{ED}$</th>
<th>$\omega_{CA}$</th>
<th>$\omega_{LS}$</th>
<th>$\omega_{DSB}$</th>
<th>$\omega^\tau$</th>
<th>variance</th>
</tr>
</thead>
<tbody>
<tr>
<td>$M - V$</td>
<td>0.5708</td>
<td>0.1438</td>
<td>0.125</td>
<td>0.1604</td>
<td>0.765</td>
<td>0.91</td>
</tr>
<tr>
<td>$M - CVaR$</td>
<td>0.4316</td>
<td>0.1165</td>
<td>0.2769</td>
<td>0.1749</td>
<td>0.765</td>
<td>1.0</td>
</tr>
<tr>
<td>$M - GP$</td>
<td>0.8127</td>
<td>0.018</td>
<td>0.0583</td>
<td>0.1110</td>
<td>0.841</td>
<td>1.31</td>
</tr>
</tbody>
</table>

Table 3(a): Optimal Hedge Fund Allocation

<table>
<thead>
<tr>
<th></th>
<th>skewness</th>
<th>Kurtosis</th>
<th>$Q_{1%}$</th>
<th>$Q_{5%}$</th>
<th>$Q_{95%}$</th>
<th>$Q_{99%}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$M - V$</td>
<td>1.89</td>
<td>11.54</td>
<td>$-2.52$</td>
<td>$-0.73$</td>
<td>2.04</td>
<td>2.5</td>
</tr>
<tr>
<td>$M - CVaR$</td>
<td>1.13</td>
<td>8.37</td>
<td>$-1.56$</td>
<td>$-0.84$</td>
<td>2.21</td>
<td>2.96</td>
</tr>
<tr>
<td>$M - GP$</td>
<td>2.89</td>
<td>21.40</td>
<td>$-2.81$</td>
<td>$-0.71$</td>
<td>2.34</td>
<td>2.89</td>
</tr>
</tbody>
</table>

Table 3(b): Optimal Hedge Fund Allocation

<table>
<thead>
<tr>
<th></th>
<th>CV$aR_{1%}$</th>
<th>CV$aR_{5%}$</th>
<th>CV$aR_{95%}$</th>
<th>CV$aR_{99%}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$M - V$</td>
<td>4.1</td>
<td>1.81</td>
<td>2.28</td>
<td>2.58</td>
</tr>
<tr>
<td>$M - CVaR$</td>
<td>3.50</td>
<td>1.8</td>
<td>2.62</td>
<td>3.51</td>
</tr>
<tr>
<td>$M - GP$</td>
<td>5.78</td>
<td>2.32</td>
<td>2.61</td>
<td>2.99</td>
</tr>
</tbody>
</table>

Table 3(c): Optimal Hedge Fund Allocation
2.55 Out-of-sample Comparison

The above analysis was based on optimal portfolio construction for FoF using all the available monthly returns. Following, I conduct an out-of-sample comparison of portfolios obtained by mean-variance, mean-CVaR and mean-GPP method over the period from January 2004 to December 2007. More specifically, I construct portfolios from the beginning of January 2004, and rebalance them semi-annually. Assuming the initial wealth of all portfolios is 1, Figure 1 presents the growth of the portfolio wealth as a function of time for the mean-variance criterion, mean-CVaR criterion and mean-GPP criterion. Clearly, the mean-GPP optimal portfolio consistently outperforms the mean-variance and mean-CVaR optimal portfolios.

Figure 2.1: Profit and Loss curves. Profit and Loss curves for portfolios obtained by the mean-variance criterion (dashed line), the mean-CVaR criterion (dotted line) and the mean-CP criterion (solid line). All portfolios are rebalanced semi-annually during January 2004 to December 2007.
2.6 Conclusions

Several recent articles on financial risk management emphasize the importance of pessimistic risk measures and promote the implementation of coherent regular risk measures in assets allocation. However, due to the lack of utility motivation, there still seems to be lack of a guideline that what kind of investors would benefit from shifting to pessimistic measures from the conventional variance or VaR measures.

In this paper, I take utility considerations seriously and uses the utility specification as a guide for choosing market risk measures and solving portfolio allocation problem. I demonstrate that that the mean-CVaR analysis, which minimizes portfolio's conditional value-at-risk for a given level of expected return, implies a highly unrealistic truncated utility function and fails to maximize most investors’ expected utility. I therefore propose a piecewise line concave utility function with endogenous reference points to capture investors’ full-range preferences over the entire distribution of portfolio returns. I show that the utility maximization problem is equivalent to minimize the general pessimistic risk measure of portfolio returns. Furthermore, quantile regression provides a robust framework to estimate the expected utility for investor with general pessimistic preferences in the presence of non-Gaussian returns of financial assets. The asymptotic properties of optimal mean-GP portfolio weights are derived.

In the empirical analysis, I consider the general pessimistic (GP) portfolio allocation problem for a fund of hedge funds (FoF) to allocate capital across different hedge fund strategies, and illustrate the flexibility and the improvement of the mean-GP criterion over the conventional mean-variance and mean-CVaR methods. Both the in-sample and out-of-sample backtesting confirm that the optimal mean_GP portfolio outperforms both the mean-variance and mean-conditional VaR portfolios by achieving higher portfolio returns.
2.7 Appendix

**Proposition 1:** Suppose the portfolio return $R$ has a distribution function, $F(\cdot)$. Then, the expected utility is

\[
E[U(W)] = \sum_{i=1}^{M+1} \left( A_i \int_{q_{\alpha_i}}^{q_{\alpha_{i-1}}} RdF(R) + B_i \int_{q_{\alpha_i-1}}^{q_{\alpha_i}} dF(R) \right)
\]

\[
= \sum_{i=1}^{M+1} A_i \left( \int_{-\infty}^{q_{\alpha_i}} RdF(R) - \int_{-\infty}^{q_{\alpha_i-1}} RdF(R) \right) + \sum_{i=1}^{M+1} B_i (\alpha_i - \alpha_{i-1})
\]

\[
= \left( \sum_{i=1}^{M+1} A_i \int_{-\infty}^{q_{\alpha_i}} RdF(R) + A_{M+1} \int_{-\infty}^{+\infty} RdF(R) \right)
- \left( \sum_{i=2}^{M+1} A_i \int_{-\infty}^{q_{\alpha_{i-1}}} RdF(R) \right) + \sum_{i=1}^{M+1} B_i (\alpha_i - \alpha_{i-1})
\]

\[
= \mu_R \sum_{i=1}^{M+1} (A_i - A_{i+1}) \int_{-\infty}^{q_{\alpha_i}} RdF(R) - \sum_{i=1}^{M} (B_{i+1} - B_i) \alpha_i - B_{M+1}
\]

\[
= \mu_R \sum_{i=1}^{M} (A_i - A_{i+1}) \alpha_i \varrho_{\alpha_i}(R) - \sum_{i=1}^{M} (B_{i+1} - B_i) \alpha_i - B_{M+1}
\]

where, $\mu_R$ is the mean of distribution of portfolio return $R$, and $\varrho_{\alpha_i}(R)$ is the $\alpha$-risk (expected shortfall) of portfolio return given $R$ is smaller than its $\alpha_i$-th quantile, $q_{\alpha_i}$.

As a result, my problem of maximizing expected utility under loss aversion is equivalent to minimization of a weighted summation of $\alpha$-risks, i.e.

\[
\sum_{i=1}^{M} (A_i - A_{i+1}) \alpha_i \varrho_{\alpha_i}(R).
\]

Moreover, the weights are determined by investor’s preference over risk and profit—the slopes of line segments of the utility function (the marginal utilities).
\[
\max_\omega E[U(\omega'r)] \Leftrightarrow \min_\omega \left[ \sum_{i=1}^M (A_i - A_{i+1})\alpha_i \bar{g}_{\alpha_i}(\omega'r) - A_{M+1}\mu_R + \sum_{i=1}^M (B_{i+1} - B_i)\alpha_i \right]
\]

With a sample of \( N \) observations from the return distribution of assets return \( r \), the optimal portfolio weights can be determined by solving the following minimization problem.

\[
\min_\omega \left[ \sum_{i=1}^M (A_i - A_{i+1})\alpha_i \bar{g}_{\alpha_i}(\omega'r) - A_{M+1}\mu_R + \sum_{i=1}^M (B_{i+1} - B_i)\alpha_i \right] \\
\Leftrightarrow \min_\omega \left[ \sum_{i=1}^M (A_i - A_{i+1})\alpha_i \left( \frac{1}{\alpha_iN} \min_{\xi_i} \sum_{t=1}^N \rho_{\alpha_i}(\omega'r_t - \xi_i) - \frac{1}{N} \sum_{t=1}^N \omega'r_t \right) - A_{M+1} \frac{1}{N} \sum_{t=1}^N \omega'r_t \right] \\
\Leftrightarrow \min_\omega \left[ \sum_{i=1}^M (A_i - A_{i+1}) \min_{\xi_i} \sum_{t=1}^N \rho_{\alpha_i}(\omega'r_t - \xi_i) - \left( \sum_{i=1}^M \alpha_i(A_i - A_{i+1}) + A_{M+1} \right) \sum_{t=1}^N \omega'r_t \right].
\]

As I assumed that the marginal utility shrinks as the level of wealth goes up, that is, \( a_{i-1} \leq a_i \), for all \( i \in [1, M+1] \). The above minimization problem is equivalent to the following

\[
\min_\omega,\xi \left[ \sum_{i=1}^M (A_i - A_{i+1}) \sum_{t=1}^N \rho_{\alpha_i}(\omega'r_t - \xi_i) - \left( \sum_{i=1}^M \alpha_i(A_i - A_{i+1}) + A_{M+1} \right) \sum_{t=1}^N \omega'r_t \right].
\]

**Propositions 2 and 3:** I first study

\[
\left( \tilde{\beta}, \tilde{\xi}(\alpha_i), i = 1, \ldots, M \right) = \arg\min_{\beta, \xi_i, i = 1, \ldots, M} \left\{ \sum_{i=1}^M \lambda_i \sum_{t=1}^N \rho_{\alpha_i} (y_t - \xi_i - \beta^T x_t) \right\}.
\]

where \( (y_t, x_t) \) corresponds to the returns or their transformations, the exact form of \( x_t \) may depend on the restrictions that I impose on the questions.
Consider the following model

\[ y_t = \xi + \beta^\top x_t + u_t \]

where

\[ Q_{yt}(\alpha_i|x_t) = \xi(\alpha_i) + \beta^\top x_t. \]

For convenience of asymptotic analysis, I introduce the following assumptions.

**Assumptions:**

1. \( u_t \) is iid with continuous CDF \( F(\cdot) \), and strictly positive density \( f(\cdot) \) at \( F^{-1}(\alpha_i) \) for \( i = 1, \ldots, M \).

2. \( D_{xx} = \lim_{n \to \infty} \frac{1}{n} \sum_{t=1}^{n} (x_t - \bar{x})(x_t - \bar{x})^\top \) is positive definite.

Let \( \psi_\tau(u) = \tau - I(u < 0) \), define

\[
G_{n1}(\xi, \beta) = \begin{bmatrix}
\frac{1}{\sqrt{n}} \sum_{t=1}^{n} \lambda_1 \psi_{\alpha_1} (y_t - \xi_1 - \beta^\top x_t) \\
\vdots \\
\frac{1}{\sqrt{n}} \sum_{t=1}^{n} \lambda_L \psi_{\alpha_L} (y_t - \xi_L - \beta^\top x_t)
\end{bmatrix}
\]

\[
G_{n2}(\xi, \beta) = \left[ \frac{1}{\sqrt{n}} \sum_{i=1}^{M} \sum_{t=1}^{n} \lambda_i \psi_{\alpha_i} (y_t - \xi_i - \beta^\top x_t) x_t \right].
\]

Then

\[
\|G_n(\beta, \bar{\xi})\| = o_p(1).
\]
I further define

\[ v_i = \sqrt{n} [\xi_i - \xi(\alpha_i)] , w = \sqrt{n} [\beta - \beta_0] \]

and rewrite \( G_{n1}(\xi, \beta), G_{n2}(\xi, \beta), G_n(\xi, \beta) \) as \( G_{n1}(v, w), G_{n2}(v, w), G_n(v, w) \), let

\[
G_0 = \begin{bmatrix} G_{01} \\ G_{02} \end{bmatrix}
\]

where

\[
G_{01} = \begin{bmatrix} \frac{1}{\sqrt{n}} \sum_{t=1}^{n} \lambda_1 \psi_{\alpha_1} (u_{t,\alpha_1}) \\ \cdot \cdot \cdot \\ \frac{1}{\sqrt{n}} \sum_{t=1}^{n} \lambda_L \psi_{\alpha_L} (u_{t,\alpha_L}) \end{bmatrix}
\]

\[
G_{02} = \frac{1}{\sqrt{n}} \sum_{i=1}^{M} \sum_{t=1}^{n} \lambda_i \psi_{\alpha_i} (u_{t,\alpha_i}) x_t
\]

and \( u_{t,\alpha_i} = y_t - Q_{y_i}(\alpha_i | x_t) \), and let

\[
L_n(v, w) = - \begin{bmatrix} A & 0 \\ 0^\top & B \end{bmatrix} \begin{bmatrix} v \\ w \end{bmatrix}
\]

where

\[
A = \text{diag} \left[ \lambda_1 f \left( F^{-1}(\alpha_1) \right), \ldots, \lambda_L f \left( F^{-1}(\alpha_L) \right) \right],
\]

\[
B = \sum_{i=1}^{M} \lambda_i f \left( F^{-1}(\alpha_i) \right) \frac{1}{n} \sum_{t=1}^{n} (x_t - \bar{x})(x_t - \bar{x})^\top,
\]
\[
v = \begin{bmatrix}
v_1 \\
\vdots \\
v_L
\end{bmatrix}
\]

then under regularity assumptions, for any \( M > 0 \),

\[
\sup_{0 \leq \|(v, w)\| \leq M} \| G_n(v, w) - G_0 - L_n(v, w) \| = o_p(1),
\]

and thus

\[
\sqrt{n} \left[ \hat{\beta} - \beta_0 \right] = B^{-1} G_{02} \Rightarrow N(0, \Sigma)
\]

where

\[
\Sigma = \frac{\sum_{i=1}^{M} \sum_{j=1}^{M} \lambda_i \lambda_j \left( \min(\alpha_i, \alpha_j) - \alpha_i \alpha_j \right)}{\sum_{i=1}^{M} \sum_{j=1}^{M} \lambda_i \lambda_j f \left( F^{-1}(\alpha_i) \right) f \left( F^{-1}(\alpha_j) \right)} D_{xx}^{-1}
\]
2.8 References


